Ramsey's Little Argument

Emerson P. Doyle

The University of Western Ontario
Department of Philosophy
edoyle8@uwo.ca

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Differing Conceptions of Logic

Recall Jean van Heijenoort (1967):

Logic as Calculus | Logic as Language

Useful framework for thinking about the development of logic, if a little coarse-grained.

Interest in Ramsey's Argument

In this same vein, I think Ramsey's argument can be taken to represent a crossroads of sorts for a variety of intersecting notions of logic and logical truth developing during the mid-twenties.

What I will focus on today is the contrast between Epistemology and Metaphysics.

The Received Wisdom

Frank Ramsey is usually noted for popularizing the observation that the logical paradoxes can be divided into two sorts.

Leads him to note that Russell's Theory of Types can likewise be distinguished into two parts.

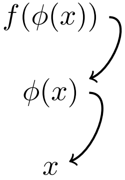
Type Theory

The Simple Hierarchy of Types stratifies the universe according to the range of significance of the arguments to a function.

This blocks the derivation of the 'mathematical paradoxes'.

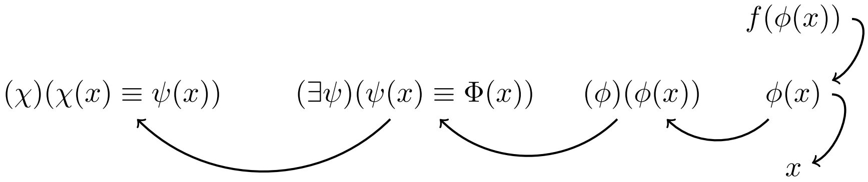
Russell's Paradox:
The class of all classes that don't contain themselves

$$\phi(\sim \phi(x))$$



Type Theory

The Ramified Hierarchy stratifies each type into an embedded hierarchy of orders according to quantificational complexity.



Required to stop derivation of the 'semantic paradoxes', like the Liar Paradox. 'False' cannot apply to everything I say at once.

Type Theory

Ramified Type Theory is far too weak a system to recover mathematics, since we cannot meaningfully make statements about all functions, all classes, or all propositions.

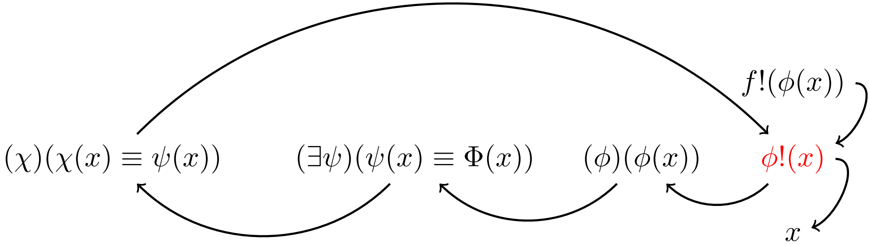
Leibniz's Law (Identity):

$$x = y =_{df} (\phi)(\phi x \supset \phi y)$$

Type Theory

Axiom of Reducibility

For any propositional function $\chi(x)$ of any order, there is an extensionally equivalent function of lowest order $\phi!(x)$.



Type Theory

Leibniz's Law

$$x = y =_{df} (\phi)(\phi!x \supset \phi!y)$$

The Received Wisdom

Ramsey: Reducibility required only when we make the mistake of supposing that properly semantic notions need be treated in what is otherwise a purely formal system.

Reducibility subject to a wide variety of other criticisms. Russell and Whitehead themselves state:

> *That the axiom of reducibility is self-evident is a proposition that can hardly be maintained. (p. 59)*

All the better then that Ramsey showed us how to avoid the whole complicated mess of ramification in the first place.

An Overlooked Argument

Such a quick summary overlooks a good deal of the insight and influence in Ramsey's few contributions (died in 1930, at only 27).

One item of importance is a nice little direct argument against reducibility being a statement of pure logic.

Ramsey's Argument

In §IV of FoM he discusses the tautological nature of Russell's two remaining contentious axioms (Infinity and Choice), adding:

> *In this inquiry I shall include from curiosity, the Axiom of Reducibility, although, since we have dispensed with it, it no longer really concerns us.*

Ramsey's Little Argument

(a) The axiom is not a contradiction, but may be true.

For it is clearly possible that there should be an atomic function defining every class of individuals. In which case every function would be equivalent not merely to a [predicative] but to an atomic function.

(b) The axiom is not a tautology, but may be false.

For it is clearly possible that there should be an infinity of atomic functions, and an individual a such that whichever atomic function we take there is another individual agreeing with a in respect of all the other functions, but not in respect of the function taken. Then $(\phi)(\phi!x \equiv \phi!a)$ could not be equivalent to any [predicative] function of x.

Ramsey's Argument

(b) The axiom is not a tautology, but may be false.

An infinity of atomic functions, and an individual a.

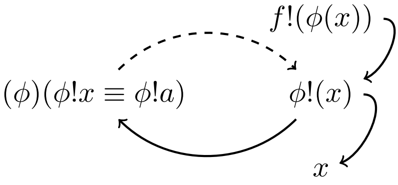
$$\{f, g, h, \ldots\} \qquad \{a, b, c, \ldots\}$$

Whichever function we take, there is another individual agreeing with a in respect of all other functions,

$$f(a) \qquad \begin{aligned} g(a) &\equiv g(c) \\ h(a) &\equiv h(c) \\ f(a) &\not\equiv f(c) \end{aligned}$$

but not in respect of the function taken.

Then $(\phi)(\phi!x \equiv \phi!a)$ could not be equivalent to any [predicative] function of x.



An Elaboration Using Classes

Consider the classes determined by our functions:

$$(\phi)(\phi!x \equiv \phi!a) \longrightarrow \{a\}$$

Since by definition for any predicative function $\phi!x$, there will be some individual that disagrees with a in respect to that function (i.e. $\phi!x \not\equiv \phi!a$ for some x).

On the other hand, every predicative function $\phi!x$ is true of more than one individual, and so will determine a wider class.

$$\phi!x \longrightarrow \{a, b\} \quad \text{(at least)}$$

Thus the higher-order function $(\phi)(\phi!x \equiv \phi!a)$ is not extensionally equivalent to any predicative function $\phi!x$.

$$\{a\} \neq \{a, b\}$$

Black's Objection

Black's (1933) survey of logic offers an objection to the argument.

The method used by [Ramsey] consists in making certain assumptions (a) concerning the number of individuals in the universe, (b) concerning the number of predicative propositional functions, and (c) the number of predicative propositional functions which are satisfied by each individual. (p. 117)

Black's Objection

> The mistake made in the proof referred to above consisted in neglecting to observe the *necessary* conditions which predicative propositional functions must obey, e.g. if f is a predicative propositional function, so is $\sim f$; if f and g are so is $h(x) = f(x) \cdot g(x)$ Df. Thus statements (a), (b), (c) above must conform to these conditions.

The complaint is just that Ramsey is forgetting that propositional functions are closed under the sentential connectives.

An Example

In a finite universe, this objection clearly holds good.

$$\{a, b, c\} \quad \{f, g\} \qquad \begin{array}{ccc} f(a) & f(b) & \sim f(c) \\ g(a) & \sim g(b) & g(c) \end{array}$$

Again, the higher-order function $(\phi)(\phi!x \equiv \phi!a)$ will denote $\{a\}$. On the other hand, $f \rightarrow \{a, b\}$ and $g \rightarrow \{a, c\}$.

Following Black though, we can construct the predicative function

$$f(x) \cdot g(x) \rightarrow \{a\}$$

But this misses the essential part of Ramsey's argument, that both the number of elements in the universe and the number of predicative functions ranging over them are infinite.

Ramsey and Russell on Logic

Black's objection runs roughshod over the very things that distinguish Ramsey and Russell's conceptions of logic:

- Ramsey allows for infinitely long formulae
- Ramsey's logic is extensional (as opposed to intensional)

These elements are what actually allow Ramsey to recover a theory of generality without the worry of falling into paradox, and so are the reason he does not need ramification.

Ramsey and Russell on Logic

For Russell, the elements of a class are determined by a propositional function.

The language is indefinitely extensible via definition, and so without ramification, the range of any given totality may be illegitimate.

For Ramsey, classes are constituted by their members, which are Witt. elementary propositions.

So while our classes may include all arbitrary subsets of individuals, this cannot lead to paradox because the members of any class are given to us from the beginning.

For Russell we cannot know what will be included in a given totality until we have constructed it; for Ramsey we do not need to know, because every totality has a constitution completely independent of its definition.

Ramsey and Russell on Logic

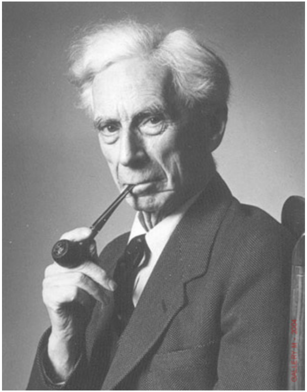
But this also eliminates a primary motivation for a distinction between functions and classes in the first place.

Ramsey and Russell on Logic

Russell's motivation:

...to enable propositions of finite complexity to deal with infinite classes of terms... (*Principles*, 1903)

...no proposition which we can apprehend can contain more than a finite number of apparent variables, on the ground that whatever we can apprehend must be of finite complexity. (*Principia*, 1910)



Ramsey and Russell on Logic

Russell is interested to explain how our finite intellect is able to 'grasp' the infinite classes of mathematics. We cannot be 'acquainted' with such classes, but similarly to his theory of descriptions, we can be acquainted with the propositional functions that determine such classes.

Ramsey's conception of logic completely disregards this epistemic angle, and so cannot tell a similar explanatory story.

Highlights a deep distinction between Russell and Ramsey's (Wittgenstein's) philosophy.

Ramsey and Russell on Logic

Russell is concerned to offer an account of our knowledge of mathematics by explaining how it is that we can have access to its objects.

So the logic serves an importantly epistemic role. The value of a propositional function is a proposition, and we must take care not to presuppose/use what we have not yet shown ourselves able to access.

Ramsey is concerned only with the logicist project—of demonstrating that mathematics is logic (tautologous).

So the logic serves an importantly metaphysical role. That we are limited beings has nothing to do with the logical facts of the world. Wittgenstein's theory is one that explains the logical constitution of the world, not our access to it.

Ramsey's Little Argument

Ramsey's argument nicely embodies the important differences between these two conceptions of logic, by demonstrating that the Axiom of Reducibility is not a tautology—a concept ultimately foreign to the *Principia Mathematica*, and to the attitude underlying that work.

Ramsey's extension of the *Principia* to infinitary formulae is at least as strong as the *Principia* with Ramification and Reducibility, but these differing assumptions highlight the attitudes and purposes to which each logic is put.

Thanks.

Friedrich Waismann (1928)

Offers an elaboration of Ramsey's argument in a short paper.

Anthony Quinton (1977) offers basically the same objection as Black.

This misses what is interesting and novel about Waismann's formulation. . .

Interpreted over $\mathbb{Q}$:

individual $=$ rational r

predicate of $r =$ class of rationals corresponding to all bounded open intervals containing r.

(i) There are infinitely many individuals and predicates is satisfied.

(ii) No two individual (rationals) satisfy all predicates (intervals).

(iii) Each predicate (interval) satisfies more than one individual (rational).

Waismann Concludes

This shows our conditions to be free from contradictions. (p. 3)

Metalogic

Question: What does it mean for the axiom of reducibility to be a tautology?

That it is a truth-function which remains true under all truth-value distributions of its arguments.

The *Principia* is a finitistic language, but this notion of a tautology rests on a truth-functional evaluation of complex propositions.

Metalogic

This brings up questions about the scope of logic:

1. We can reason about a formal system.
2. We can interpret or translate a formal system into another.

And questions about content:

3. The propositions of logic have content.
4. The propositions of logic are schematic.

Russell thought 3, and perhaps 1.

Wittgenstein thought 4, and certainly not 1 or 2.

Ramsey and Waismann? Seemingly 4.
And for Waismann at least, 2.

Carnap

Recent work by Steve Awodey and A.W. Carus on Carnap's intellectual development in the twenties provides an interesting amalgam of these ideas.

With a well-defined notion of logical consequence, the notion of a tautology becomes an analytic truth.

Instead, Carnap used a version of Simple Type Theory as a 'basic system', within which he defines the notion of an axiomatic system.

The sentences of the basic system are fully-interpreted (have a content) from the beginning, much like Russell's conception of logic.

Carnap

We take the primitive symbols of an axiom system as variables, and then write the whole axiom system as a single conjunction $f(R) = g(R) \cdot h(R) \cdot i(R) \cdot \ldots \cdot m(R)$, where predicates are axioms, and R is an n-tuple of our variables.

A function $k(R)$ is then a consequence of $f(R)$ just in case:

$$(R)(f(R) \supset k(R))$$

is true in the basic system.

A model of f is an n-tuple of logical constants from the basic system which satisfy f—in other words, $f(n)$ holds.

Carnap

This provides us with a coherent notion of tautology in the context of a finite language, by relying on all substitution instances.

In this context, Ramsey's argument holds good, and the Axiom of Reducibility is not a tautology.

So Carnap can be seen to bridge the distance between the two concepts of logic we have been discussing:

> We utilize a metaphysically loaded basic system to ground our epistemic explanatory story of how axiomatic systems can be conventional, yet still applicable to the world.

Thanks Again!