The ω-Rule and the Methodological Role of Tolerance in Carnap's *LSL*

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Thesis:

The Principle of Tolerance is entailed by Carnap's methodological analysis of the formal sciences, and so is not the foundation of, but rather plays only a narrow role in, Carnap's doctrines.

In contrast to my interpretation here, the **Deflationary View** risks turning Carnap's position into a strong relativism, and makes it otherwise rather empty and philosophically uninteresting.

Since I think there is something of value to be extracted from Carnap's work (in philosophy, logic, mathematics), I hope that an alternate reconstruction of his views brings these items to light.

Outline

1. Preliminary Remarks
    - Introduction
    - Sketch of Carnap's Philosophy
2. Some Immediate Problems
    - Gödel's Theorems
    - Tolerance and Strong Meta-languages
    - Syntactical Rules and Empirical Content
3. Deflationary View Responds
    - Deflationary Responses are Uninteresting and Unnatural
4. Why is Carnap Interested in Determinateness?
    - Logicism and Truth
    - Gödel's Theorems (Again!)
    - A Criterion for Logicality
5. Alternative Responses

The Motivating Problem

Empiricism

Sense experience or observation is the only source of our substantive knowledge about facts.

Problem

What about mathematical knowledge? This seems substantive, but not empirical. Even worse, it seems *a priori* and *necessary*.

Solution

Knowledge of mathematics is *not* substantive knowledge, it's *a priority* and necessity are grounded in logic. Logic in turn is *Conventional*—it is the inferential residue of the tacit (or explicit) syntactical rules of our language.

Logico-Mathematical Sentences

"In material interpretation, an analytic sentence is absolutely true whatever the empirical facts may be. Hence, it does not state anything about facts." (§14)

"[. . .]the mathematico-logical are analytic, with no real content, and are merely formal auxiliaries." (Foreward)

Theses About Logico-Mathematical Sentences:

1. Logico-mathematical sentences are *contentless*.
2. Logico-mathematical sentences are *syntax of language*.

Consequences—Logical Pluralism:

- All language forms are acceptable.
- Epistemic notions (truth, justification, ontology) are relativised to a particular language/framework.

Logical Pluralism

The Principle of Tolerance

It is not our business to set up prohibitions, but to arrive at conventions. [...] *In logic, there are no morals.* Everyone is at liberty to build up his own logic, i.e., his own form of language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments. (§17)

The Logical Syntax of Language (1934)

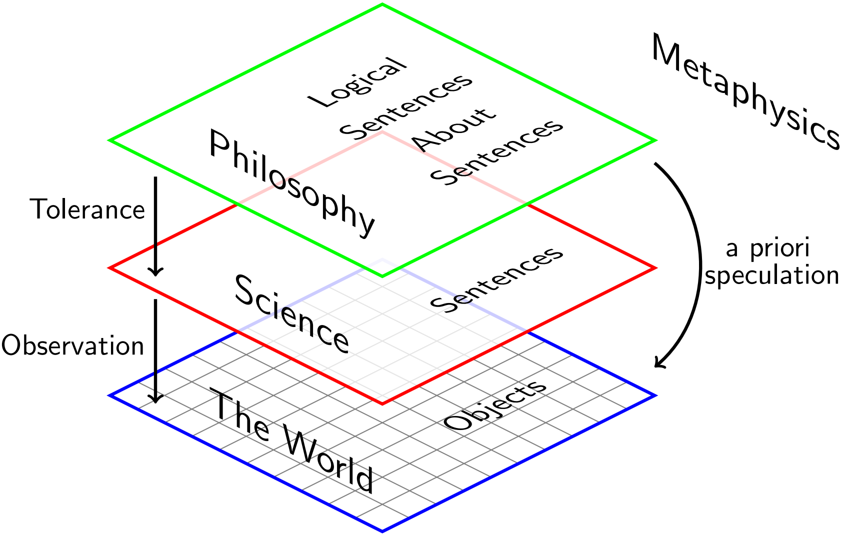
Purpose:

"The book itself makes an attempt to provide, in the form of an exact syntactical method, the necessary tools for working out the problems of the logic of science."

Conclusion:

"*Philosophy is to be replaced by the logic of science*—that is to say, by the logical analysis of the concepts and sentences of the sciences, for *the logic of science is nothing other than the logical syntax of the language of science*."

A Picture of Our Theoretical Landscape



2. Some Immediate Problems

"One of the chief tasks of the logical foundations of mathematics is to set up a formal criterion of validity, that is, to state the necessary and sufficient conditions which a sentence must fulfil in order to be valid (correct, true) in the sense understood in classical mathematics." (§34a)

Reasons? Epistemic, Ontological, Methodological, etc.

Carnap's Reason: If our task is to study the logic of science, then our syntactical investigations had better be able to recover (explicate) the notion of mathematical validity in the language of science.

Gödel's Incompleteness Theorems

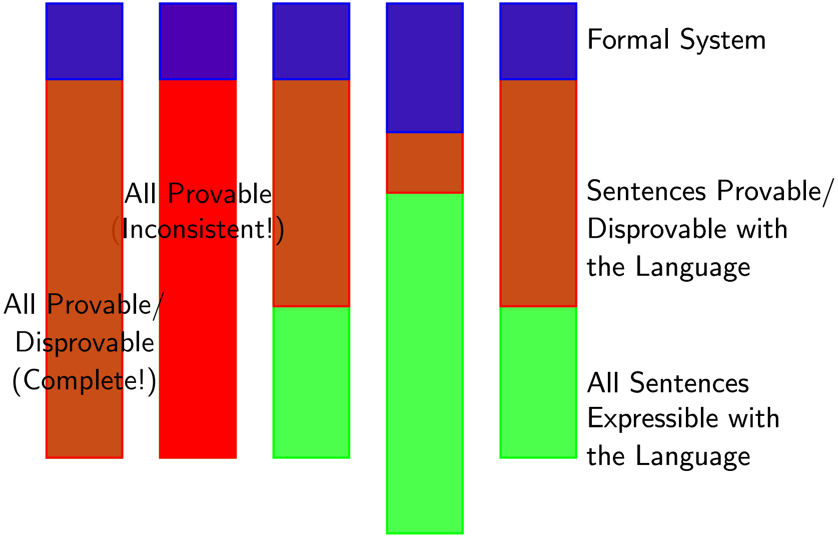
Gödel 1

For any interesting formal system, there will be sentences expressible in the vocabulary of the system which are neither provable nor refutable within that system. (Formal systems are incomplete)

Gödel 2

No interesting formal system can prove its own consistency. (A proof of consistency means inconsistency!)

The problem is with the finite notion of *Proof*:



Language I (PRA)

Rules of Derivation

Regular logical rules of inference

Finite number of premises

Provable sentences are *Demonstrable*

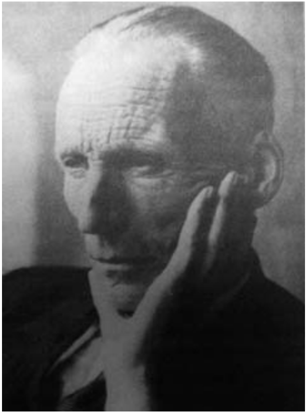
Rules of Consequence (Meta-Language)

ω-Rule

Infinite number of premises

Determinate sentences are *Analytic*

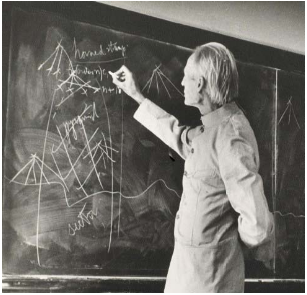
Mathematical Intuitionism



- Mathematics is the act of constructing mathematical objects in the mind, based on our pure intuition of time.
- Reject any methods of proof that do not end in the presentation of a well-defined, constructed object.
- Thus reject most infinitary notions.
- Formalization and logic are a hindrance to mathematical activity.

Carnapian Response

Once the fact is realized that all the pros and cons of Intuitionist discussions are concerned with the forms of a calculus, questions will no longer be put in the form: "What *is* this or that like?" but instead we shall ask: "How *do we wish to arrange* this or that in the language to be constructed?" or, from the theoretical standpoint: "What consequences will ensue if we construct a language in this or that way?" (§17)

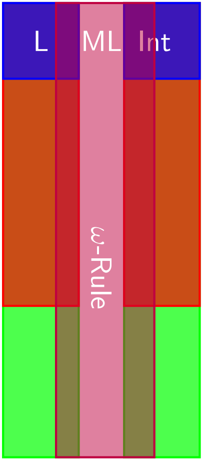


Problem:

Carnap! You suggest we resolve our disputes by displaying clearly, and then surveying and comparing, our respective languages. But to do this requires a shared meta-language stronger than either object-language. And that's my point—I reject such strong mathematical notions as nonsense!

The need for strong meta-languages to investigate the consequences of certain syntactical rules seems in conflict with the Principle of Tolerance.

Insisting on strong meta-languages so that we can be conventionalists at the object-level just begs the question against more conservative logicians.



Gödel's Argument

Recall that logico-mathematical sentences are contentless, and that they follow from the syntactical rules of a framework.

But how can we be sure that a given syntactical rule has no content? (i.e., doesn't entail any factual sentences) And so is properly logical?

A proof of the consistency of that rule would demonstrate it doesn't impinge on any factual sentences.

Gödel's Argument

Problem:

Carnap! By Gödel 2 no interesting system can prove its own consistency. Therefore, you need to ascend to a stronger meta-language to show a rule consistent. But here you will need to assume the very rule that you're going to show is consistent! So you cannot justify the thesis that mathematics is syntax of language.

Gödel's Incompleteness Theorem

Set up a formal criterion of validity in the face of Gödel 1. This corresponds to what we mean when we say: "This sentence follows (logically) from that one".

Tolerance and Meta-Languages

The Principle of Tolerance suggests resolving disputes in the philosophy of mathematics syntactically. But this requires mathematical assumptions in the meta-language that will inevitably beg the question against someone in the dispute.

Syntactic Rules and Empirical Content

For mathematics to be syntax of language, syntactical rules must not have empirical content. We can only demonstrate this with a consistency proof in a meta-language. But this requires assuming the very rules we are trying to show syntactical.

3. Deflationary Responses

Recall the Principle of Tolerance:

> *It is not our business to set up prohibitions, but to arrive at conventions.* [...] *In logic, there are no morals.* Everyone is at liberty to build up his own logic, i.e., his own form of language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments. (§17)

This suggests a *relativisation* of all empirical notions to a particular language or framework:

Thomas Ricketts (2008):

The logical consequence relation of a calculus defines standards for the acceptance and rejection of sentences and theories formulated within the calculus and defines standards for a language-relative notion of cognitive correctness. (p. 206)

And again. . .

Here we have Carnap's leading idea: the Principle of Tolerance and an attendant sharp contrast between the adoption of a formal language as the language of science and the evaluation of sentences within that language as correct or incorrect. Carnap marks this contrast [. . .] by calling the former a matter of convention. (Ibid)

The problem is that a resolution of foundational disputes by ascending to the meta-level is not forthcoming because of the very strength of the mathematics required at the meta-level begs the question against the Intuitionist.

But if we take Tolerance as the guiding maxim of Carnap's doctrine, we might suggest that Tolerance is a proposal to put aside irresolvable metaphysical/foundational debates.

Richardson (1994):

[I]f Carnap is insisting that there is no issue over which the classical and intuitionist mathematician are fighting, then he may well feel free to adopt ever stronger metalanguages in the course of syntactic investigations. This won't beg any question against the intuitionist, as there is no question here to be begged. Tolerance can then be seen as an invitation to set aside endless pseudo-disputes. (p. 73)

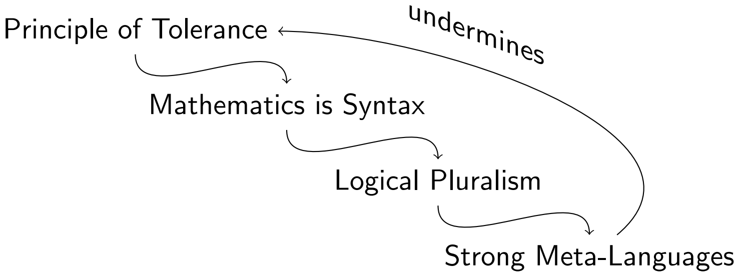
So we can see Carnap as re-orienting the philosophy of mathematics in such a way that foundational disputes are dissolved, in a similar way to the way he suggests metaphysical disputes be dissolved in general philosophy.

But there's still a problem here.

Even if the intuitionist agrees to Tolerance at the meta-level (she won't), there's still the question of the strength of the meta-language. Again, Richardson says:

> [I]f a strong meta language is required for the explicit presentation of the syntax of L, then certain features of the syntax language will, in this investigation, still be left implicit. But if acquiescence in nonprecise syntax language is admissible, then we seem to lose the point of the reconstructive project[...] (p. 73)

The Argumentative Picture



The deflationary view simply has to bite this bullet:

Goldfarb (2009):

[H]ow does convention get to be determined? There seems to be no way to do this, except to say that it's a matter of the meta-meta language. The nature of that language is then settled only given the nature of the meta-meta-meta language. And so on. This is not an incoherent position; it is, as I have written elsewhere, 'self-supporting at each level'. But it does have more than a whiff of circularity or at least of vacuity, which, of course, Carnap's critics will exploit. (p. 120)

Gödel's argument relies on a key assumption: a *language transcendent* notion of *Empirical Fact*. Mathematics can only impinge on factual sentences if they're already present to get in the way when we decide to lay down syntactical rules for our language.

But Goldfarb (1996) denies such a notion:

> However, as the Principle of Tolerance indicates, it is central to the metaphysics of *Logical Syntax* that any such language transcendence be rejected. Rather, the notion of empirical fact is given *by way* of the distinction between what follows from the rules of a particular language and what does not, *so that different languages establish different domains of fact*. In this way, Carnap undercuts the very formulation of Gödel's argument. (p. 227)

What of the Principle of Tolerance?

Traditional Logical Empiricist Gambit:

What is the epistemological status of the Principle of Tolerance? It's clearly not empirical, so is it conventional? If so, isn't that blatantly circular?

Ricketts (1994):

Like the principle of empiricism, the principle of tolerance itself is not a thesis, but a proposal, the expression of an attitude or standpoint. (p. 196)

And again. . .

Goldfarb (1997):

What of the Principle itself? Can Carnap be tolerant about Tolerance? The question evidences a misconstrual of the status of the Principle. [. . .] It is an exhortation to or an expression of an attitude toward languages, namely, the attitude of logical pluralism. In a sense, it is not said, but only shown. (p. 61)

Extreme Relativism?

If the domain of facts is entirely determined by the syntactical rules of a language, then the question as to what facts there are seems to become a matter of conventional choice. In other words, it seems by fiat that we can determine the facts of a matter *in just the same way* that we determine whether or not to employ the axiom of choice in our reasonings about said facts.

Without any constitutive basis whatsoever for the comparison of frameworks, it is unclear how to make any decision at all between them, pragmatic or rational.

Carnaps view on this interpretation thus amounts to the construction of isolated and independent linguistic frameworks that fail to impinge upon or contact the world in any sense.

Quitting Philosophy of Mathematics

Ricketts (2008):

We can now appreciate the deflationary character of Carnap's philosophy of mathematics. [. . .] Carnap thus does not present in *Logical Syntax* an account of the nature of mathematics, of our knowledge of mathematics, and of the applications of mathematics in empirical science comparable to the accounts developed by Kant, Mill, Frege, Wittgenstein, and Hilbert. Carnap rejects the questions these thinkers address. In a sense, he gives up philosophy of mathematics. (p. 211)

Summing Up (Carnap is Sad)

Taking a Tolerance-First approach to Carnap's philosophy does get us out of some trouble. And into some other difficulties:

Extreme Relativism
We get out of Gödel's argument, but *Empirical Fact* is language-relative.

Tolerance is (Almost) Self-Undermining
We respond to Brouwer, but Tolerance conflicts with Carnap's goals of explicitness.

Retreat from Philosophy of Mathematics
Carnap is not answering any traditional questions in the philosophy of mathematics.

Everything is a Proposal
Carnap is not even asserting Tolerance as a thesis.

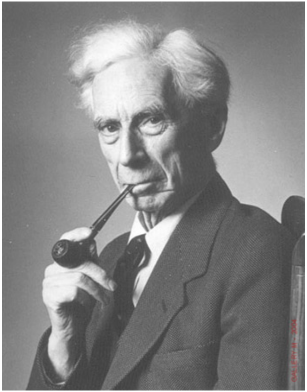
4. Determinateness

Question: Why is Carnap so interested in all logico-mathematical sentences being *Determinate*?

Recall that the Rules of Consequence for a language guarantee that every logico-mathematical sentence in the language will come out either *Analytic* or *Contradictory*.

Factual sentences will then be *Indeterminate*, or *Synthetic*.

At Least Two Reasons



The original idea of a Frege-Russell style logicism was to show that all mathematical truths are ‘reducible’ to logical truths.

There are several benefits:

1. Epistemic: foundational security
2. Methodological: security in reasoning
3. Semantic: account for our knowledge of mathematics

Gödel 1

For any interesting formal system, there will be sentences expressible in the vocabulary of the system which are neither provable nor refutable within that system.

Another reason one might be interested in having all logico-mathematical sentences be determinate is as a means to provide a well-specified account of *mathematical truth* for a given mathematical theory.

We need to do this within a semantic metalanguage for the theory.

Still gives us a well-specified semantic account of what it means for any particular logico-mathematical sentence to be true, and provides a formal explication of the concept of truth.

Carnap has no interest in trying to specify a truth-definition for mathematical languages, and says as much in §60b:

> *For truth and falsehood are not proper syntactical properties*; whether a sentence is true or false cannot generally be seen by its design, that is to say, by the kinds and serial orders of its symbols.

Yet Carnap still goes through the trouble of proving, for each language he considers in *Logical Syntax*, that every logico-mathematical sentence is determinate, and all synthetic sentences are indeterminate.

In fact, it's more complex than this: He designs the ω-Rule just to get around Gödel 1, and ensure that every logical sentence of Language I is determinate. He sets it up that way.

The answer arrives in *General Syntax*, §50, where Carnap is trying to lay out a general theory for any language forms whatsoever:

> [I]f we reflect that all the connections between logico-mathematical terms are independent of extra linguistic factors, such as, for instance, empirical observations, and that they must be solely and completely determined by the transformation rules of the language *we find the formally expressible distinguishing peculiarity of logical symbols and expressions to consist in the fact that each sentence constructed solely from them is determinate*,

So Determinateness acts as a *criterion for logicality*—it is the unique formal feature that logical sentences share.

But why *Determinateness*?

Recall that Carnap asserts (1) That logical sentences have no content; and (2) That logical sentences are formal auxiliaries.

These theses can only be maintained if logical sentences follow from the rules of a framework, and this is exactly what the notion of determinateness captures.

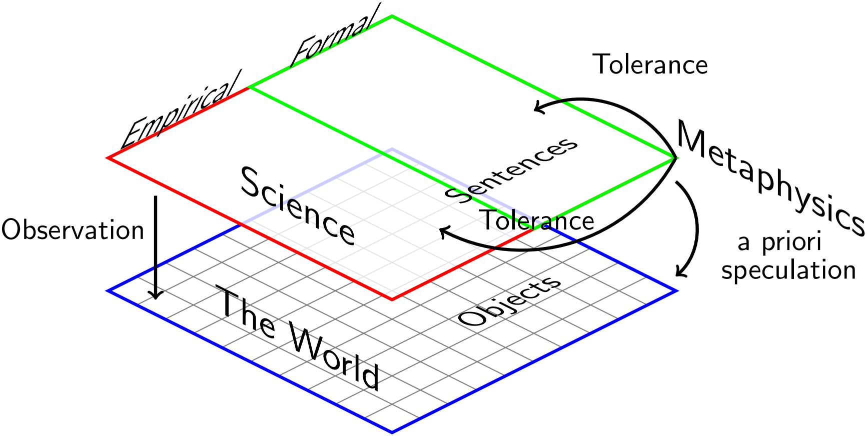
The ω-Rule then, acts to show that Carnap's condition of adequacy is met for Language I. Rather than just stipulating a consequence relation, he gives an informative analysis of why it should be the way it is, and then proves the he meets his analysis.

But note that this analysis of logico-mathematical sentences is *antecedent* to the Principle of Tolerance.

It is these characteristics of logical sentences which allow them to submit to Tolerance. Tolerance is the premier methodological tool of the *Formal*, as opposed to the *Empirical* sciences.

> The tolerant attitude here suggested is [. . .] tacitly shared by the majority of mathematicians. (§17)

The Logical Syntax of Language



Methodological Analysis of Science

Logic and Mathematics is Syntax (Determinate)

Principle of Tolerance

Logical Pluralism

Carnap's methodological analysis of the formal sciences is what *motivates* and *licenses* the use of Tolerance in choosing a language for the reconstruction of science. Further, Carnap's analysis of philosophy shows it to be likewise formal, and that's why Tolerance is applicable to philosophical disputes: they're about language.

> [T]he term 'consequence' is the only one that exactly corresponds to what we mean when we say: "This sentence follows (logically) from that one", or: "If this sentence is true, then (on logical grounds) that one is also true." (§14)

In §34h, Carnap proves that the Principle of Mathematical Induction and the Axiom of Choice are analytic. But he blatantly assumes these principles in the meta-language for his proof. So what's the point?

> [The proofs] only show that our definition of 'Analytic' effects on this point what it is intended to effect, namely, the characterization of a sentence as analytic if, in material interpretation, it is regarded as logically valid.

5. Alternative Responses

Responding to Gödel is actually easy.

Recall:
For mathematics to be syntax of language, syntactical rules must not have empirical content. We can only demonstrate this with a consistency proof in a meta-language. But this requires assuming the very rules we are trying to show syntactical.

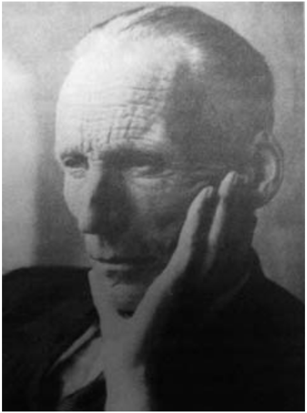
But Gödel, we don't need to demonstrate that the syntactical rules have no empirical content, it merely needs to be the case that they don't. Gödel's argument is plainly unsound. (Awodey & Carus, 2004)

But Gödel's argument does bring up a good point:

Carnap's goal is a reconstruction of the total language of science for the purposes of conceptual clarification and discovery. If his methodological analysis of the sciences is taken as the basis for his work, then our languages *should* be beholden to some extra-linguistic notion.

At least, the analytic/synthetic distinction needs to be effected in such a way that this methodological analysis is respected.

We *can* construct a language in such a way that most of mathematics ends up synthetic (Carnap notes this), but such a language is more than just less useful—it actually *gets things wrong* in the sense that it misses the key characteristics that make each science what it is.



This leads us to a response to Brouwer.

Recall:
The Principle of Tolerance suggests resolving disputes in the philosophy of mathematics syntactically. But this requires mathematical assumptions in the meta-language that will inevitably beg the question against someone in the dispute.

But Brouwer, I'm not basing my argument on Tolerance, I'm basing it on the methodological analysis of science!

Why is This Better?

Carnap (1939):

Concerning mathematics as a pure calculus there are no sharp controversies. These arise as soon as mathematics is dealt with as a system of "knowledge" [. . .] Now, if we regard interpreted mathematics as an instrument of deduction within the field of empirical knowledge rather than as a system of information, then many of the controversial problems are recognized as being questions not of truth but of technical expedience. (p. 50)

The point is that, as far as the empirical sciences go, we will use whatever mathematics is most expedient (this Brouwer can't argue with). What matters is consistency—all other worries in mathematics can be handled by Tolerance, precisely because it's a formal science.

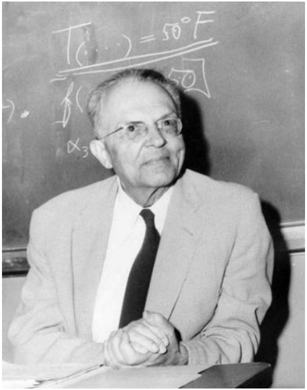
But What About the Circularity?

What circularity? What Carnap has done is given an informative *analysis* of the methods of science, and mathematics' place in them. We identified a criterion of adequacy for that analysis (Determinateness), and shown that our analysis meets it (ω-Rule). That we must rely on stronger mathematics in order to carry through this analysis is less important now, because we understand that what we have is an *explication* of the methodology of formal science.

So the whole thing doesn't have to be self-supporting.

Conclusions

1. I take it that this view of Carnap's philosophy, with the methodological analysis of science coming prior to Tolerance, falls naturally out our discussion of Carnap's use of the ω-Rule.
2. The position also responds to both Brouwer and G"odel.
   - Gödel's argument fails, but it shows us that our reconstructions must be tied to the world in a particular way.
   - Brouwer's complaint is handled by a re-orientation of the philosophy of mathematics, but it has nothing to do with Tolerance.
3. Most importantly, Carnap is addressing many questions in the philosophy of mathematics: we gain an account of applicability, an understanding of logical knowledge, and an account of mathematical truth (there isn't any).



Thanks!