Ramsey's Little Argument*

Emerson P. Doyle

In his seminal paper 'Logic as Calculus and Logic as Language', Jean van Heijenoort examines the methodological implications of a distinction made by Frege to contrast his system of the *Begriffsschrift* from that of Boole's *Laws of Thought*. Frege characterizes his system as not merely a *calculus ratiocinator*—or as a simple algebra of unanalyzed propositions—but also as a *lingua characterica*—a universal language capable of expressing and analyzing the meaning and content of complex scientific propositions. Van Heijenoort argues that understanding what Frege meant by this distinction will garner useful insight into the history of logic.

Another distinction that can be identified in the history is one not internal to logic itself, but rather involves the methodological role that logic plays in relation to philosophy. The distinction I have in mind is that between an understanding of logic as constrained by epistemology, and the notion that logic act as a guide to metaphysics. We might think of this as a distinction in how logic is used, or perhaps more fundamentally, what logic *is* in a sense. As with Frege's distinction above, this opposition is meant to be neither exclusive nor exhaustive. And again as above, I think that an examination of its methodological implications can likewise offer useful insights into the history of logic. At least, this is what I shall argue.

Unlike Frege's discussion however, to the best of my knowledge no one in the history makes explicit mention of the logic as metaphysics/epistemology distinction. Still there are examples which do well to bring the distinction into focus, such as an oft-overlooked argument by Frank Ramsey to the effect that Russell's Axiom of Reducibility is not a tautology. Ramsey's 'The Foundations of Mathematics' (FoM) is a lengthy attempt to diagnose and then resolve various perceived issues with Russell and Whitehead's logistic

*For conversations and comments on earlier drafts of this paper, all of which led to substantial improvements, I would like to thank Steve Bland, William Demopoulos, Sona Ghosh, and Robert Moir. Of course any errors remain my own.

program in their *Principia Mathematica*[1]. The article is perhaps best known for dividing the foundational paradoxes into two classes[2]: those that involve logical notions essentially, like Russell's paradox, and those, like the paradox of the liar, which involve "some reference to thought, language, or symbolism, which are not formal but empirical terms." (FoM, p. 183) Given this observation, Ramsey goes on to isolate a simple hierarchy of types as distinct from the ramified hierarchy, arguing that because we can effectively ignore 'empirical terms' in a purely formal endeavour, this simplified system suffices for the desired logicist reduction. The motivation for this simplification is to avoid the need for reducibility, required in Russell and Whitehead's case to strengthen their ramified theory, but objectionable on the grounds both that it is non-obvious, and that it has significant existential import and so is a non-logical presupposition.

As described, Ramsey's strategy is indirect: by showing that there is no need for reducibility in the first place, he avoids having to engage in a discussion as to the logical nature of the axiom. Even given this overarching structure, Ramsey still goes out of his way to provide an additional direct argument against reducibility. In a section on the tautological nature of *Principia's* remaining 'contentious axioms' (Choice and Infinity), he asserts that "[i]n this inquiry I shall include from curiosity, the Axiom of Reducibility, although, since we have dispensed with it, it no longer really concerns us." (FoM, p. 220) Ramsey's comments make the argument seem as almost an afterthought, but with the benefit of hindsight, we can today see it as nicely situated at a crossroads for a wide range of methodological trends that arose during the mid-twenties—including the distinction between logic as a metaphysical guide and logic as constrained by epistemology.

The plan is to proceed as follows. We will begin by reviewing the ramified theory of types, within which the axiom of reducibility plays a key role. We will then outline and discuss Ramsey's direct argument against the axiom. In the following sections I will place this argument in the context of what I take to be reasonable interpretations of both Ramsey and Russell's logical methodology. Our focus will be their interpretations of the notion of a propositional function, and their understanding of logical truth. Ramsey's argument will thus serve as a kind of case study by providing a focus for the distinction in logical methodology I wish to highlight.

[1]All references to the *Principia* are from the 1997 abridged edition.

[2]The original insight is due to Peano, to whom Ramsey gives due credit.

1 Ramified Type Theory

Our discussion of ramified type theory will diverge significantly from some more formal presentations of the same, such as Church (1976) and Myhill (1974). The reasons for this are three. First, we will for the most part confine ourselves to that fragment of the *Principia's* language involving only monadic functions whose argument is restricted to the level immediately lower than that of its function. The inclusion of polyadic functions, which may be relations between arguments of distinct (but always lower) levels, would greatly increase the complexity of our exposition. Secondly, our discussion should be general enough to accommodate any reasonable development of ramified type theory, and so will not depend on the peculiarities of some particular formalization. Finally, our presentation will reflect Ramsey's above-mentioned technical insight that a 'simple' hierarchy of types can be distinguished from the more complex ramified hierarchy.

In his own presentation of ramified types, Ramsey observes that the development of type theory was motivated by the discovery of antinomies such as Russell's paradox and the paradox of the liar.

> These contradictions it was proposed to remove by what is called the Theory of Types, which consists really of two distinct parts directed respectively against the two groups of contradictions. These two parts were unified by being both deduced in a rather sloppy way from the 'vicious-circle principle', but it seems to me essential to consider them separately. (FoM, p. 187)

Russell, in line with Poincaré, was convinced that a unified solution to all the paradoxes was preferable to a piecemeal one. In the *Principia*, after enumerating a series of seven well-known antinomies, Russell asserts:

> In all of the above contradictions (which are merely selections from an indefinite number) there is a common characteristic, which we may describe as self-reference or reflexiveness. [...] In each contradiction something is said about *all* cases of some kind, and from what is said a new case seems to be generated, which both is and is not of the same kind as the case of which *all* were concerned in what was said. (p. 61–62, original emphasis.)

Even though logicians today distinguish the paradoxes along the same lines as Ramsey, it is important to recognize that Russell's understanding and application of the vicious-circle principle in developing ramified types is actually a matter of some subtlety. Following Gödel ([1944]1983), we find two distinct sorts of principles which Russell asserts in the *Principia*, but tends to conflate by referring to them as merely differing formulations of the same principle.[3] The first is a general constraint on our ability to consider certain collections as "totalities". A characteristic statement in the *Principia* being: "Given any set of objects such that, if we suppose the set to have a total, it will contain members which presuppose this total, then such a set cannot have a total." (p. 37) The second formulation is a prohibition on impredicative definition, which Russell expresses most clearly when he says: "...there must be no totalities which, if legitimate, would contain members defined in terms of themselves." (p. 160)

Like Gödel twenty years later and contra Russell, Ramsey advances a realist stance on classes and so argues that the prohibition effected by this second formulation of the vicious-circle principle is unjustified. This is just because there is nothing vicious in the process of characterizing some object in terms of the group of which it is a member. This only becomes a problem when we instead attempt to *construct* an object in such a way, since we would then be presupposing that which we are in the process of constructing. It is this sort of construction that the vicious-circle principle is meant to forbid. The methodological implications of Ramsey's manoeuvre in his argument here will be our focus. For the moment however, we note merely that where Ramsey stands correct is in the observation that this second formulation of the vicious circle principle involving impredicativity plays no part in the resolution of the logical paradoxes. Here the simple hierarchy of types is entirely sufficient, as we will now see.

A *type* is taken by Russell to be the range of significance of a propositional function. More explicitly, the legitimate arguments to a function make up a range for which the function is said to be significant. Functions are then stratified into a hierarchy according to the types of their arguments, while at the lowest level we have the type of individuals. We then proceed to functions of individuals, functions of functions of individuals, and so on up the hierarchy indefinitely. Russell's notation distinguishes between denoting a function $\phi\hat{x}$, and denoting any *value* of that function, ϕx. We shall do

[3] For further discussion of the relation between the differing forms of the vicious-circle principles, see Demopoulos (2007). My interpretation of Russell throughout is heavily indebted to this paper.

this also. Observe now that since no function can be meaningfully taken as an argument to itself, the construction of the Russell class necessary in the derivation of Russell's paradox is blocked. Such is the simple hierarchy of types. The number of elements in the type of individuals is assumed by Russell to be infinite, as embodied in the *Principia's* axiom of infinity.[4]

Now to ramification, which imposes a further hierarchical structure upon the functions at each type, except the first, into a hierarchy of *orders*. In keeping with the generality of our discussion, the levels of this hierarchy may be either stratified or arranged cumulatively. The important point to observe is the fact that the hierarchy is dependent upon the quantificational structure of said functions, and this is where a restriction upon certain kinds of *definition* becomes important. Consider the propositional function defined $\phi\hat{x} =_{df} (\psi)(\psi\hat{x} \supset \psi a)$, which is obviously a function of x and so a function of type 1 (assuming individuals are of type 0). Quantification over the function $\psi\hat{x}$ here presupposes all functions of type 1. However, we are trying to *define* a function of type 1, and so we have violated the impredicative form of the vicious-circle principle by presupposing that which we are attempting to define. Thus we must further restrict the range of functions *within* each type on the basis of their quantificational complexity and definition. Functions of the lowest order in any given type are called *predicative functions*, denoted $\phi!\hat{x}$, and have no bound variables. Second-order functions presuppose the given totality of functions of first-order, those of third-order presupposing totalities of second and first-order, and so on, indefinitely up the hierarchy within each type. We would then offer as our earlier definition the second-order function $\phi_2\hat{x} =_{df} (\psi)(\psi!\hat{x} \supset \psi!a)$.

Ramification does indeed solve the non-logical paradoxes (while Ramsey calls them 'empirical', we might do better to call them the 'semantic' paradoxes) by denying us the ability to construct certain illegitimate totalities. Consider for example Berry's paradox of *the smallest natural number not*

[4] There is some question as to whether, given this assumption, the *Principia* still constitutes a *logicist* project as traditionally conceived—certainly Frege would not have been happy with such an assumption. Cf. Boolos ([1994]1998) for discussion on this point. He concludes that while the *Principia* may satisfy the traditional logicist in terms of defining all the concepts of mathematics logically; owing to infinity, it does not satisfy the further requirement that we derive all the propositions of mathematics without some non-logical residue. This is Ramsey's criticism also, but as regards reducibility. A question I will examine below is whether or not Russell was actually attempting to satisfy the latter condition—the answer is not straightforward.

nameable in under eleven words. The paradox is of course that the italicized phrase names just this smallest natural number, but since the phrase is only ten words long, that number must not be the said smallest natural number so nameable. The resolution is to note that notion of 'nameability' is here ambiguous with respect to order. In other words, we note that the names for any given object of type τ will be of type $\tau + 1$, for which ramification will serve to divide those names into a hierarchy of names of differing order. When well-specified then, the problem becomes *the smallest natural number not nameable with a name of order ω in under seventeen words*. Now we can indeed prove that the number named in the second italicized expression is nameable in under seventeen words, but *not* by a name of order ω—according to the theory of types, the expression itself is a name of order $\omega + 1$. Thus the paradox is resolved.

As is well known, the unfortunate consequence of ramification is that vast portions of mathematics become unrecoverable in the theory of types. This owes to the fact that much mathematics involves impredicative constructions. Take for example Russell and Whitehead's preferred definition of identity, Leibniz's Law: $x = y =_{df} (\phi)(\phi x \supset \phi y)$, which quantifies over all functions of a particular type. Just as with the notion of 'nameable' above, the functions of any given type will be divided by order according to their quantificational complexity, and so such quantification over *all* functions of a type becomes illegitimate. In other words, with the introduction of orders the variable ϕ no longer has the requisite scope, since its range will be restricted to functions of some particular order. Thus, we cannot say that two individuals are identical when they share *all* their properties, as needed. The introduction of axioms of reducibility[5] are therefore required to imbue the system with the necessary strength by asserting at every type the existence of, for every propositional function of any order, an extensionally equivalent propositional function of lowest order. A simple instance of the reducibility schema then alleviates our difficulty with Leibniz's Law by postulating equivalent predicative functions for functions of any higher order. In symbols $(\exists\psi)(x)(\phi x \equiv \psi!x)$. Thus the axiom recovers our ability to capture all functions of a given type with a single quantifier by asserting the existence of a complete set of extensionally equivalent predicative functions for the purposes

[5] For ease of presentation I often speak of '*the* axiom of reducibility'. It should be understood that there are in fact an infinite collection of such axioms, one such scheme for functions of each arity, at each order, of every type (except the lowest).

of expressing generalities. Identity thus becomes $x = y =_{df} (\phi)(\phi!x \supset \phi!y)$, where (ϕ) quantifies over all predicative functions. Similarly mathematical induction, transfinite arithmetic, and the development of analysis rest in the *Principia* on the use of reducibility.

Although noted by most commentators, it bears repeating that the axiom of reducibility does not reinstate the semantic paradoxes, even though at first blush it seems as though it may.[6] This owes to the *Principia's* being an *intensional* logic. In other words, propositional functions are taken in the *Principia* to be abstract, intensional entities. They can neither be identified with the open sentences of the language (and so are not nominalistic), nor identified with the classes which they determine. This is obvious given the ramification scheme, since the functions $\phi\hat{x}$ and $(\exists x)(\phi x \cdot \phi\hat{y})$ may determine the same class but are of different orders. Thus while the axiom of reducibility posits a non-denumerable number of extensionally distinct propositional functions of lowest order, a simple cardinality argument shows that it will not be the case that all such functions will be definable in the language. So while there will be a function of lowest order that corresponds to some higher-order function needed to legitimately determine some totality, in any case where paradox may arise there is no way to construct paradoxical propositions referring to said totality at the required order to produce an antinomy. The axiom thus provides the requisite scope to our quantifiers by assuming the existence of certain functions without also installing the means to construct impredicative functions in the language.

This intensional character of propositional functions is in contrast to an extensional conception of classes, wherein classes are determined entirely by their members, rather than the members being determined by some defining characteristic which all the elements share. It is therefore important to recognize that the notion of class at play in the *Principia* is not intensional. While the members of a class are certainly determined by a propositional function (perhaps by several), classes themselves are still constituted by their members. As Russell explains:

> ...every propositional function about a class expresses an extensional property of the determining function of the class, and therefore does not depend for its truth or falsehood upon the particular function selected for determining the class, but only upon

[6]For further discussion on this point and a more complete elaboration of type theory in general, see Copi (1971).

the extension of the determining function. (p. 191)

The hierarchy of classes thus constitutes a simple hierarchy corresponding to the simple hierarchy of predicative functions described above. This does not however commit Russell and Whitehead to the outright existence of classes, since reference to classes in the propositions of the *Principia* are always eliminable through the device of contextual analysis.[7] While many commentators have placed the focus of their interpretation upon this ontological 'reduction' of classes to propositional functions, in practice the *Principia* actually remains agnostic as to the existence of classes. In fact, our interpretation below will instead focus on the epistemological work that propositional functions do in providing an explanation of our knowledge of classes. I will argue that such an interpretation remains in the spirit of Russell's broader epistemological methodology. In contrast to this, Ramsey's reformulation of the *Principia* amounts to an eschewal of the careful distinction between classes and propositional functions, or a wholehearted acceptance of the existence of classes as independent extensional entities. Just how this is done will also be discussed below.

2 Ramsey's Little Argument

As is well known, the axiom of reducibility was met with almost immediate criticism. Russell and Whitehead themselves anticipated such objections, admitting: "That the axiom of reducibility is self-evident is a proposition which can hardly be maintained." (*Principia*, p. 59) And again in the introduction to the second edition of 1925, Russell writes:

> One point in regard to which improvement is obviously desirable is the axiom of reducibility (*12.1.11). This axiom has a purely pragmatic justification: it leads to the desired results, and to no others. But clearly it is not the sort of axiom with which we can rest content. (p. xiv)

We have briefly noted above Ramsey's argument to the effect that reducibility is required only when we make the mistake of supposing that properly

[7]Cf. Demopoulos (2007). Briefly, we can always eliminate reference to classes in a proposition by reformulating the proposition into one about a class' corresponding propositional function. I will discuss the method of contextual analysis further below.

semantic notions need be treated in what is otherwise a purely "symbolic system" (FoM, p. 184). Ramsey's further argument aims to show that reducibility is not a logical truth, and so has no business assumed as an axiom of Russell's formal system.

Here is the text of Ramsey's argument, in full:

> (a) The axiom is not a contradiction, but may be true.
>
> For it is clearly possible that there should be an atomic function defining every class of individuals. In which case every function would be equivalent not merely to a [predicative] but to an atomic function.
>
> (b) The axiom is not a tautology, but may be false.
>
> For it is clearly possible that there should be an infinity of atomic functions, and an individual a such that whichever atomic function we take there is another individual agreeing with a in respect of all the other functions, but not in respect of the function taken. Then $(\phi)(\phi!x \equiv \phi!a)$ could not be equivalent to any [predicative] function of x. (FoM, p. 220)

The first point to notice is Ramsey's use of the notion of an *atomic function*, absent from the 1910 *Principia.* Adhering closely to Wittgenstein's doctrine, Ramsey explains that an atomic proposition is one which involves no logical operators, and so is completely logically simple, such as ϕa. What follows is the now-standard truth-functional account of propositional logic, but extended into a functional calculus by noting that atomic functions express propositions when names (of individuals and properties) are substituted for variables. As expected, a *tautology* is a proposition which agrees with all truth-possibilities, while a contradiction agrees with none. Again following Wittgenstein, where Ramsey diverges from contemporary practice is in extending this analysis to allow for infinite truth-functions. Note that $(x)(\phi x)$ and $(\exists x)(\phi x)$ are propositions, since what they assert is simply the logical product or sum of the set of all propositions $\phi\hat{x}$. "Thus general propositions containing 'all' and 'some' are found to be truth-functions, for which the arguments are given in another way." (FoM, p. 171) Ramsey thereby concludes (although he says this is "hard to prove" (p. 172)) that all propositions are truth-functions of atomic propositions.

The argument itself is relatively straightforward. Part (a) assumes that the full powerset of a domain is available, or in other words that every subset of individuals is determined by a legitimate function. Given a denumerable domain as with the axiom of infinity, this results in a non-denumerable number of functions defined on that domain. Ramsey notes explicitly that his reinterpretation of the notion of a function captures a wider range of functions than is originally possible in the *Principia*, since even "indefinable" (FoM, p. 186) classes and relations are now captured. We will have more to say of this reinterpretation below, for now I will merely point out that the main differences are Ramsey's aforementioned use of an extensional understanding of classes in tandem with the Wittgensteinian notion of infinite truth-functions. These ideas together have strong metaphysical consequences, and suggest abandoning the impredicative form of the vicious-circle principle as unduly restrictive, and so with it the axiom of reducibility.

Part (b) is, for the moment, our primary concern. The argument purports to show that, in the context of the *Principia's* system of ramified types, not every higher-order function is extensionally equivalent to some predicative function. Since this is just the assumption that the axiom of reducibility asserts, a counterexample demonstrates its non-tautologousness. We can perhaps see the argument a little more clearly in terms of the classes determined by Ramsey's exemplar propositional functions. $(\phi)(\phi!x \equiv \phi!a)$ will determine the singleton $\{a\}$, since for any predicative function $\phi!\hat{x}$, by construction $\phi!x \not\equiv \phi!a$ for some individual x. Thus, *only* a will satisfy the equivalence, and we have our singleton. But as regards the strictly predicative functions, none will determine the class $\{a\}$, since again by construction every predicate is true not only of a, but also of some other individual. Thus the higher-order function $(\phi)(\phi!x \equiv \phi!a)$ is not equivalent to any predicative function in this universe, and so the axiom of reducibility fails.

2.1 An Objection by Max Black

It will be beneficial to pause briefly and review an objection to Ramsey's line of reasoning here offered by Max Black (1933).[8] Quoting Black:

> The method used by [Ramsey] consists in making certain assumptions (a) concerning the number of individuals in the universe, (b)

[8]Anthony Quinton (1977) proposes what amounts to the same objection to a more perspicuous version of the argument offered a few years later by Fredrich Waismann ([1928]1977). While the details are historically interesting, considerations of space preclude me from dealing explicitly with this formulation of the argument and its consequences.

concerning the number of predicative propositional functions, and (c) the number of predicative propositional functions which are satisfied by each individual. (p. 117)

I have no immediate objection to any of these conditions. He continues by summarizing the method of Ramsey's argument:

> If, in such a universe a non-predicative propositional function can be constructed and shown to be equivalent to no predicative propositional function, the axiom of reducibility would be false in that domain. (*Ibid.*)

Finally, here is Ramsey's error according to Black:

> The mistake made in the proofs referred to above consisted in neglecting to observe the *necessary* conditions which predicative propositional functions must obey, e.g. if f is a predicative propositional function, so is $\sim f$; if f and g are so is $h(x) = f(x) \cdot g(x)$ Df. Thus statements (a), (b), (c) above must conform to these conditions. (*Ibid*, emphasis in original.)

The objection then is basically that Ramsey's argument steps outside the confines of the *Principia's* conditions on acceptable functions. These 'conditions' are analogous to the standard formation rules for the quantifier-free fragment of first order logic. More explicitly, I believe Black has in mind the idea that a conjunction of functions is also a function.

In a *finite* universe, this objection clearly holds good against Ramsey. Consider for example a universe with three objects $\{a, b, c\}$ and two predicates $\{f, g\}$, where $f(a), f(b), \sim f(c), g(a), \sim g(b)$, and $g(c)$ hold true. Such a situation meets all of Ramsey's conditions (besides the infinity of individuals and functions of course). As with Ramsey's argument, we can again specify a higher-order function $(\phi)(\phi!x \equiv \psi!a)$, which will again denote the class $\{a\}$. It is easy to see that neither of our predicative functions f, g will do the same, since $f(x)$ denotes the class $\{a, b\}$ and $g(x)$ the class $\{a, c\}$. Black's point seems to be that this is to forget that $f(x) \cdot g(x)$ is also a predicative function in the language of the *Principia*, and this *is* extensionally equivalent to our higher-order function. But this misses the essential part of Ramsey's argument: that both the number of elements in the universe and the number of predicative functions ranging over them are infinite. Given this, we would need an infinitely long conjunction of predicative functions to distinguish Ramsey's singleton class $\{a\}$. As expected however, infinite formulae are barred in the language of the *Principia*, and so given any domain of interest for reconstructing mathematics (i.e. a domain with an infinity of individuals) and the assumption of a complement of atomic functions ranging over that domain in the way Ramsey specifies in his impromptu theory, Black's objection seems to fail.

We can be more explicit in our analysis of Black's objection by following Michael Potter—the only other contemporary discussion of this argument I have found—and expressing Ramsey's argument in a more contemporary form.[9] Take our domain to be the natural numbers, Ramsey's distinguished individual a to be 0, and suppose his infinite collection of atomic functions are of the form $\hat{x} = n$ for $n \geq 1$. Considering all possible truth functions of our predicative functions (and so meeting Black's objection), it is easy to see that any predicative function true of 0 will be true of all but finitely many numbers; while any predicative function false of 0 will be false of all but finitely many numbers. Take for example the predicative function $\sim (\hat{x} = 1 \vee \hat{x} = 2)$, this will be true of 0 and all other numbers except 1 and 2. On the other hand, the predicative function $(\hat{x} = 1 \vee \hat{x} = 2)$ will be false of 0 and all other numbers except 1 and 2. Thus, the higher-order function $(\phi)(\phi!x \equiv \phi!0)$ will be true of only 0, but it will not be extensionally equivalent to any finite predicative truth-function.

What Potter effectively does is to provide a model in which the conditions of Ramsey's argument hold good, or in other words we have provided a consistency proof of the theory by interpreting it in the natural numbers. The model thus acts as a counterexample to the idea that the axiom of reducibility is satisfied in all models of the theory, which is the generally accepted notion of a logical truth. Furthermore, in this case it is more obvious that the standard truth-functional formation rules are observed. Contra Black's objection then, the counterexample turns on our ability to specify a particular set of functions defined on a particular domain, wherein the axiom fails, while logical truths—tautologies in Ramsey's sense—of course continue to hold good. Thus the axiom is not a tautology.

[9]Cf. Potter (2002, p. 161).

3 Ramsey on Predicative Functions

This reflection suggests a further question as to just how Ramsey's conception of logical truth differs from that in the *Principia*. In the introductory section to FoM, he explains that his motivating criticism of the *Principia* is its focus on providing a logical account of the *concepts* of mathematics to the detriment of a robust account of its *propositions*:

> Here there are really two distinct categories of things of which an account must be given—the ideas or concepts of mathematics, and the propositions of mathematics. This distinction is neither artificial nor unnecessary, for the great majority of writers on the subject have concentrated their attention on the explanation of one or the other of these categories, and erroneously supposed that a satisfactory explanation of the other would immediately follow. (FoM, p. 165)

For Wittgenstein and Ramsey an account of logical propositions as tautologies suggests a particular understanding of the logical constitution of the world. As mentioned above, atomic proposition are composed of simples—these amount to the basic elements and qualities inherent in the world. Differing combinations of these simples comprise the differing truth-possibilities, of which propositions (atomic and truth-functional) express agreement and disagreement according to the circumstances of the actual world. While both Wittgenstein and Ramsey are notoriously unclear about just what these simples are, they nonetheless stand as a significant assumption underlying this account of propositions, and so Ramsey's account of logical truth.

The upshot of the above account is that it suggests to Ramsey a reinterpretation of Russell's notion of a predicative function as divorced from its intensional character inherent in the *Principia*.

> A *predicative function* of individuals is one which is any truth-function of arguments which, whether finite or infinite in number, are all either atomic functions of individuals or propositions. (FoM, p. 202)

What is important to note is that for Ramsey, *both* the domain of individuals and the domain of atomic propositions are 'completed totalities', determined prior to our expression or construction of them. This entails that the range of all atomic functions (which are just the atomic propositions) is likewise given to us on account of the logical constitution of the world. In essence what Ramsey does is to present an alternate means by which to individuate propositional functions, one that does not rely on their mode of construction.

> Admitting an infinite number [of arguments] involves that we do not define the range of functions as those which could be constructed in a certain way, but determine them by a description of their meanings. They are to be truth-functions—not explicitly in their appearance, but in their significance—of atomic functions and propositions. (*Ibid.*)

According to Ramsey then, functions are to be individuated by their sense or meaning, which is taken to be the truth-tables of the propositions which are the function's values. Normally, we construct a propositional function by replacing some name in a propositional symbol with a variable. In the case of functions of individuals, this is straightforward, since the domain of individuals forms an objective and 'completed totality'. In the case of functions of functions however, we must be more careful owing to the semantic paradoxes. Since the propositional symbol we mean to replace by a variable may itself contain a quantifier, this would entail the creation of a new functional symbol which may lie in the domain over which we have already quantified. Given Russell's notion of propositional function in the *Principia*, this sort of construction is obviously ruled out by the impredicative form of the vicious-circle principle. By maintaining that the range of all atomic propositions is also a 'completed totality', and asserting that all propositions are truth-functions of said atomic propositions on the other hand, Ramsey removes the risk that we might illegitimately construct a symbol which relies upon itself to determine its range in a paradoxical way, and thus eschews the need for the axiom of reducibility.

Another way to see this is to recall our previous analysis of Black's objection to Ramsey's argument against reducibility. By allowing for an infinitary language by which to express generality, we remove the dependence upon functions of higher order to express certain propositions. While we still require the symbolic device of the universal quantifier to express such propositions, for Ramsey this owes entirely to the poverty of our abilities of expression—a contingent matter—and is not something logically relevant.[10]

[10]Cf. FoM, p. 204–205. Ramsey here gives his well-known example of specifying the tallest person of a group with reference to the entire group itself. There is certainly nothing

What underlies such expressions is an infinitary truth-function composed of fully-realized simples, and so the worry of impredicatively defining some function whose domain we have not yet specified falls away.

As an example, recall the functions $\phi\hat{x}$ and $(\exists x)(\phi x \cdot \phi\hat{y})$. In the *Principia* these constitute functions of differing orders, even though they may determine the same class, or in other words are extensionally equivalent. We can now note with Ramsey however that this notion of an order is in fact a property of the *symbol* used to express the function, not of the function itself. Given the current interpretation, these two distinct symbols are but different ways of designating the same set of atomic propositions which are themselves composed of an objective set of logically simple elements and qualities, rather than distinct intensional entities which just so happen to determine the same class. The result is of course a wholly extensional view of logic. We might say that for Ramsey, the logical space of propositional functions is given to us as a completed totality pre-theoretically, while for Russell, it is our ability to potentially construct certain functions in the language which determines the overall logical space of propositional functions.

3.1 Extensionality and the Semantic Paradoxes

Before moving on, we should pause once again to ascertain explicitly whether Ramsey's interpretation of the notion of a propositional function reinstates the semantic paradoxes. In other words, we can ask: If it is the symbolism of the *Principia's* language which leads to the semantic paradoxes, by collapsing the hierarchy of orders and removing the axiom of reducibility, does Ramsey not prime the language of his reinterpreted system with the ability to once again generate semantic paradoxes?

The quick answer is that he does not, and we already have an explanation as to why his system is not so susceptible. Rather than an intensional logic, Ramsey's logic is extensional—a logic of classes:

> I do not use the word 'class' to imply a principle of classification, as the word naturally suggests, but by a 'class' I mean any set of things of the same logical type. Such a set, it seems to me, may or may not be definable either by enumeration or as the extension

vicious in such a construction, and he argues that the case of impredicative definition in mathematics is analogous. Given his assumptions we have enumerated about the logical constitution of the world, this seems to be the case.

> of a predicate. If it is not so definable we cannot mention it by itself, but only deal with it by implication in propositions about all classes or some classes. (FoM, p. 178)

Perhaps more than any other, this passage betrays Ramsey's realist convictions. The subject of some propositions—certain classes—may or may not be accessible to us, but they remain the subjects of said propositions just the same. He concludes that mathematics is "essentially extensional, and may be called a calculus of extensions, since its propositions assert relations between extensions." (FoM p. 178)

We can see extensionality as a conscious restriction on the scope of Ramsey's logic in contrast to that in the *Principia*. A consequence of this is expressed by Ramsey's overarching strategy regarding the paradoxes that we reviewed briefly in the introduction. Simply, by so restricting his logic, Ramsey overtly excludes the ability to express the problematic semantic notions in the formal apparatus of his language. Instead, these informal notions must be relegated to an external metalanguage. Once so dismissed, a solution to the now informal semantic paradoxes proceeds in a way very much like Russell's ramification scheme. Ramsey's idea is to note that semantic notions such as 'means', 'designates', or 'nameability' are ambiguous, and so must be relativized to a hierarchy of languages if their analysis is to be treated in any sort of rigorous way. Along similar lines to Russell's (1922) suggestion of a hierarchy of languages in his introduction to the *Tractatus* as a means to overcome Wittgenstein's insistence that nothing can be properly said of the world as a whole but can merely be shown, we can see Ramsey's suggestion here as an even more explicit—albeit small—step toward the eventual distinction between syntax and semantics.[11]

Given our analysis in this section, it seems entirely accurate for Ramsey to characterize his program in FoM as an attempt to "...reduce a calculus of extensions to a calculus of truth-functions" (FoM, p. 177). Recognizing mathematics as extensional, Ramsey bars any mention of intensional or semantic entities within his system. He then offers a characterization of logic as truth-functional, and so his project becomes one of reducing the extensional science of mathematics to a system of tautologies. In order to carry

[11] I do not by this comment mean to imply anything so strong as that Ramsey here anticipates Tarski's work on semantics and model theory. On the other hand, the recognition that certain semantic notions cannot be adequately treated within the object system was certainly 'in the air' at the time.

through this very traditional logicist program, we have seen that Ramsey makes several metaphysical assumptions about the inherent structure of the world which underlie his doctrine of propositions, and so his understanding of the nature of logical truth. These ideas then suggest a reinterpretation of the notion of a propositional function, an interpretation again guided by certain attitudes toward logic as a means for the analysis of the logical constitution of the world, at least as regards a logicist project. In the final section, we will contrast this attitude toward logic with an interpretation which casts Russell's attitude toward the same as decidedly more epistemic.

4 Russell's Theory of Classes

For our purposes, the main consequence of Ramsey's reinterpretation of the *Principia* is the destruction of Russell's careful distinction between propositional functions and classes inherent in his intensional logic. Recall that because the elements of a class are determined by propositional functions for Russell, there is no need in the *Principia* to posit the independent, objective existence of a hierarchy of classes:

> It is not necessary for our purposes to assert dogmatically that there are no such things as classes. It is only necessary for us to show that the incomplete symbols which we introduce as representative of classes yield all the propositions for the sake of which classes might be thought essential. When this has been shown, the mere principle of economy of primitive ideas leads to the non-introduction of classes except as incomplete symbols. (*Principia*, p. 72)

While we have seen that Ramsey's manoeuvre allows him to dispense with ramification and the axiom of reducibility, it also eliminates a primary motivation for Russell's distinction in the first place.

For the sake of comparison, recall first that Frege ([1884]1980) frames his foundational investigations as motivated by his famous question, in §62 of the *Grundlagen*: "How then, are the numbers to be given to us, if we cannot have any ideas or intuitions of them?" His answer is of course that our knowledge of number is provided by extensions of concepts, or what we have been calling an intensional conception of classes as determined by propositional functions. As Demopoulos (2007) notes, we may pose as a motivating factor for Russell's logicism an attempt to answer the further but analogous question of explaining how classes are 'given' to us without appeal to ideas or intuition. The problem is exacerbated when, following Russell, we recognize that knowledge of an infinite class cannot consist in the knowledge of each of its members individually.

As early as *The Principles of Mathematics* ([1903]1996) we see in Russell's theory of denoting concepts an attempt at an answer to these questions:

> Indeed it may be said that the logical purpose which is served by the theory of denoting is, to enable propositions of finite complexity to deal with infinite classes of terms [...] Now, for my part, I see no possible way of deciding whether propositions of infinite complexity are possible or not; but this at least is clear, that all the propositions known to us (and, it would seem, all propositions that we *can* know) are of finite complexity. It is only by obtaining such propositions about infinite classes that we are enabled to deal with infinity; and it is a remarkable and fortunate fact that this method is successful. (§141, original emphasis)

Russell thus argues that our knowledge of the infinite must be mediated through the finite. In this case, denoting concepts act as the constituents of a proposition which encapsulate this relation and allow us to grasp classes of infinite complexity as the subject of a proposition by finite means. By the time of the *Principia* the answers are much the same, but now the device of propositional functions has come to act as the mediator to explain our grasp of the infinite:

> ... a [propositional] function can be apprehended without its being necessary to apprehend its values severally and individually. If this were not the case, no function could be apprehended at all, since the number of values (true and false) of a function is necessarily infinite and there are necessarily possible arguments with which we are unacquainted. (pp. 39-40)

Again, we see that some sort of mediating entity is necessary to explain our ability to grasp propositions whose subjects may be of infinite complexity or are otherwise beyond our acquaintance.

The notion of *acquaintance* should be familiar from Russell's theory of descriptions, wherein knowledge of the subject of a proposition of which we are not acquainted is explained by contextual analysis.[12] To take the classic example, we are not acquainted with the subject of the proposition 'The present King of France is bald.' on account of there being no such subject. Rather, this proposition can be analyzed into an existence claim about *some* unique object which satisfies such-and-such properties. In this case there is of course no such subject, and so we recognize the proposition as false. This judgement is only possible given that we can understand the proposition in the first place, which for Russell means being in a special epistemic relationship with its constituents. The method of contextual analysis thus provides an explanation for this understanding in this special case where there is no constituent to be so acquainted.

But as Demopoulos (2007) notes, the resolution of the semantic issues surrounding vacuous descriptions is not the only lesson to be taken from the theory of descriptions as related to the theory of classes in the *Principia*. The theory of descriptions acts as an explanation for our knowledge of things in any case where we are not directly acquainted with the subject of a proposition. For our purposes this is especially relevant in the case of a general proposition of which we cannot be acquainted with all of its instances, because there are infinitely many for example. Similarly, the use of variables as constituents of propositional functions allow them to denote ambiguously all of their instances. We grasp the propositional function itself, and through this mediated relation gain epistemic access to its totality of values, even if we are not acquainted with said values—in fact even if we cannot be. The passage of the *Principia* quoted directly above makes this parallel between descriptive and propositional functions clear.

Russell's understanding of predicative functions as epistemically unobjectionable thus acts as a means to ground this *epistemological* theory regarding our knowledge of classes, and so provides an answer to the questions motivating his logicism. The intensional nature of Russell's logic, and with it the logical notion of class, is thus of paramount importance, since our knowledge of a class is explained from above by our acquaintance with a propositional function which determines it. Given this interpretation, rather than follow-

[12]See Russell (1912) for an outline of Russell's epistemological views as to knowledge we have by acquaintance versus knowledge by description. These are both species of our knowledge of *things*, as opposed to our knowledge of *truths*. Briefly, we can be acquainted only with things of which we are directly aware: sense-data, universals, and the self are common examples. Thus, while we can be acquainted with the colour of a table, we can know the table itself only by description involving things we know by acquaintance.

ing Ramsey and taking the impredicative form of the vicious-circle principle as an undue restriction on our ability to construct propositional functions, we can instead see it as a methodological principle to ensure that we remain honest in the explanation of our epistemological access to classes. Were we to violate the vicious-circle principle, we might then define some function with reference to a totality whose epistemic access we have not yet guaranteed. Seen in this light, the axiom of reducibility is a theoretical assumption utilized to ensure a sufficient number of predicative functions in order to secure our ability to grasp certain classes which are otherwise only accessible by functions of higher order.[13]

> A class for which we lack a predicative function is accessible only if it is determined by a logical construction built from known predicative functions. In the case of classes, the relevant transparency of the components of the functions by which they are known is their *logical* transparency, the fact that the basic functional constituents are predicative functions. In analogy with the theory of knowledge of things which transcend our acquaintance, although the basic component propositional functions are predicative, the class determined by the logical construction which they comprise can be one that is not known by means of a predicative function. (pp. 175–176, original emphasis)

As Demopoulos here explains, ramification guarantees that any higher order function will be at root “built” only of immediately accessible predicative functions, which we take to be logically transparent—we might say functions with which we can be acquainted. Given this interpretation then, the principal interest of the *Principia* for Russell is the epistemological one of explaining our knowledge of classes through the assumption of the existence of some intensional mediator—namely predicative propositional functions.

[13]Given the discussed theoretical context for the axiom of reducibility, we can observe that similarly to the axiom of infinity as discussed by Boolos ([1994]1998) and in note 4 above, Russell’s project may have been less than a traditional logicist’s one. This however is not to imply that it was in any way a failure. Indeed, the *Principia* can be taken as a successful epistemic account of our knowledge of mathematical objects as logical objects.

5 Ramsey, Russell, and Logical Methodology

These considerations lead us finally back to Ramsey's argument against the axiom of reducibility, and to the notion of a tautology which is its keystone. As is well known, Russell had a significant difficulty in coming to understand Wittgenstein's notion of a tautology. As late as 1920s' *Introduction to Mathematical Philosophy*, we find Russell asserting his inability to fully express the quality of tautologousness which, over and above extreme generality, is peculiar to logical propositions:

> ... [Tautologousness], combined with the fact that [logical truths] can be expressed wholly in terms of variables and logical constants (a logical constant being something which remains constant in a proposition even when all its constituents are changed)
> —will give the definition of logic or pure mathematics. For the moment, I do not know how to define "tautology." (p. 205)

Regardless as to why this might be the case, we have seen that Ramsey's understanding of tautology is ultimately foreign to the theoretical context of the *Principia*. Ramsey's little argument certainly demonstrates that the axiom of reducibility is not a tautology and so is entirely out of place in what I have called a 'traditional' logicist program like Ramsey's, which seeks to effect a reduction of mathematics to logic. But such a program was not necessarily Russell's.

We should keep in mind that Russell's understanding of logical truths at the time of the *Principia* was as those propositions which are maximally general. While this is inadequate as an account of logical truth (as Ramsey notes), I think it betrays Russell's methodological desire to treat mathematics as a science like any other, albeit the most abstract and general science.

> We tend to believe the premises because we can see that their consequences are true, instead of believing the consequences because we know the premises to be true. But the inferring of premises from consequences is the essence of induction; thus the method in investigating the principles of mathematics is really an inductive method, and is substantially the same as the method of discovering general laws in any other science. (Russell, [1907]1973, p. 273)

With this understanding of Russell's attitude toward the justification of mathematical premises, his defence of the axiom of reducibility in the *Principia* (quoted in part above) looks much less out of place. At the same time it becomes even easier to read his interests as primarily epistemological, attempting to provide both an account of our mathematical knowledge on the basis of our logical knowledge, and a theory as to how we might have access to the objects of said knowledge. The finitary character of the language in the *Principia* is therefore of principal importance to Russell's epistemological project of securing *our* knowledge of mathematical objects on the basis of logical objects. Russell's remarks quoted above from §141 of the *Principles* makes this clear. Whether or not there might be propositions of infinite complexity, epistemologically we seem barred from grasping them directly. Some further explanation is therefore required, and logic provides with an account of predicative functions and the method of contextual analysis .

On the other hand, Ramsey's infinitary notion of a tautology is the basis for the metaphysical or ontological task of reducing mathematics to logic simpliciter. His Wittgensteinian assumptions therefore place logic in the methodological position of providing an account of the logical constitution of the world, rather than working in tandem with an epistemic position to supply a theory about our access to it. Ramsey's little argument thus acts to embody the important differences between the methodological roles which Russell and Ramsey each take their logical systems to play in the context of their broader philosophical systems. Perhaps more importantly, the argument acts to highlight the difference in what each conceives logic to *be* in a fundamental way.

Our distinction is perhaps easiest to see in comparison of each logician's interpretation of predicative functions, as discussed above. For Russell, predicative functions work to ground a theory about our knowledge of classes by positioning them as epistemically secure entities with which we are immediately acquainted. So we might say that Russell's logic is responsible to his empiricist epistemology. Alternately, Ramsey conceives of predicative functions as composed of the basic logical constituents in the world, and serving as a means to guarantee a full compliment of logical machinery by which to effect a logicist reduction, regardless as to our ability to transparently grasp this machinery. So it is Ramsey's prior metaphysical theory of logical truth which shapes his logic. Both projects can be seen as worthwhile, but they express a definite methodological distinction in attitude toward logic and its use in scientific investigation.

References

Black, M. (1933). *The Nature of Mathematics: A Critical Survey.* London: Routledge & Kegan Paul.

Boolos, G. ([1994]1998). The Advantages of Honest Toil over Theft. In: R.C. Jeffrey (Ed.), *Logic, Logic, and Logic*, chap. 16. Harvard University Press, pp. 255–274.

Church, A. (1976). Comparison of Russell's Resolution of the Semantical Antinomies with That of Tarski. *Journal of Symbolic Logic*, 41: 747–760.

Copi, I.M. (1971). *The Theory of Logical Types.* London: Routledge & Kegan Paul.

Demopoulos, W. (2007). The 1910 Principia's Theory of Functions and Classes and the Theory of Descriptions. *European Journal of Analytic Philosophy*, 3(2): 159–178.

Frege, G. ([1884]1980). *The Foundations of Arithmetic.* Northwestern University Press. J.L. Austin (Trans.).

Gödel, K. ([1944]1983). Russell's Mathematical Logic. In: P. Benacerraf & H. Putnam (Eds.), *Philosophy of Mathematics: Selected Readings.* CUP, second edn., pp. 447–469.

Myhill, J. (1974). The Undefinability of the Set of Natural Numbers in the Ramified *Principia*. In: G. Nakhnikian (Ed.), *Bertrand Russell's Philosophy.* London: Duckworth, pp. 19–27.

Potter, M. (2002). *Reason's Nearest Kin.* OUP.

Quinton, A. (1977). Introduction. In: B. McGuiness (Ed.), *Friedrich Waismann: Philosophical Papers*, Vienna Circle Collection. D. Reidel Publishing Co., pp. ix–xx.

Ramsey, F.P. ([1925]1990). The Foundations of Mathematics. In: D.H. Mellor (Ed.), *F.P. Ramsey: Philosophical Papers.* CUP, pp. 164–224.

Russell, B. ([1903]1996). *The Principles of Mathematics.* W.W. Norton & Co., second edn.

——— ([1907]1973). The Regressive Method of Discovering the Premises of Mathematics. In: D. Lackey (Ed.), *Essays in Analysis*. London: George Allen & Unwin, pp. 272–283.

——— (1912). *The Problems of Philosophy*. Indianapolis: Hackett Publishing Co.

——— (1920). *Introduction to Mathematical Philosophy*. Dover Publications Inc.

——— (1922). *Tractatus Logico-Philosophicus*, chap. Introduction. London: Routledge, pp. ix–xxv.

van Heijenoort, J. (1967). Logic as Calculus and Logic as Language. *Synthese*, 17: 324–330.

Waismann, F. ([1928]1977). The Nature of the Axiom of Reducibility. In: B. McGuinness (Ed.), *Friedrich Waismann: Philosophical Papers*, Vienna Circle Collection. D. Reidel Publishing Co., pp. 1–3.

Whitehead, A.N. & Russell, B. (1997). *Principia Mathematica to *56*. CUP, abridged edn.