An Objection to Intuitionism

Emerson Doyle
The University of Western Ontario
CSHPM: June 1st, 2008

George Francois Cornelis Griss

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- Offers an informal reconstruction of mathematics without the use of negative properties.
- F.O. Intuitionistic Arithmetic (HA) demonstrated recoverable in Negationless Arithmetic (NA).
- I contend that Griss' objection remains both historically *and* philosophically interesting.

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- Discuss Griss' objection.

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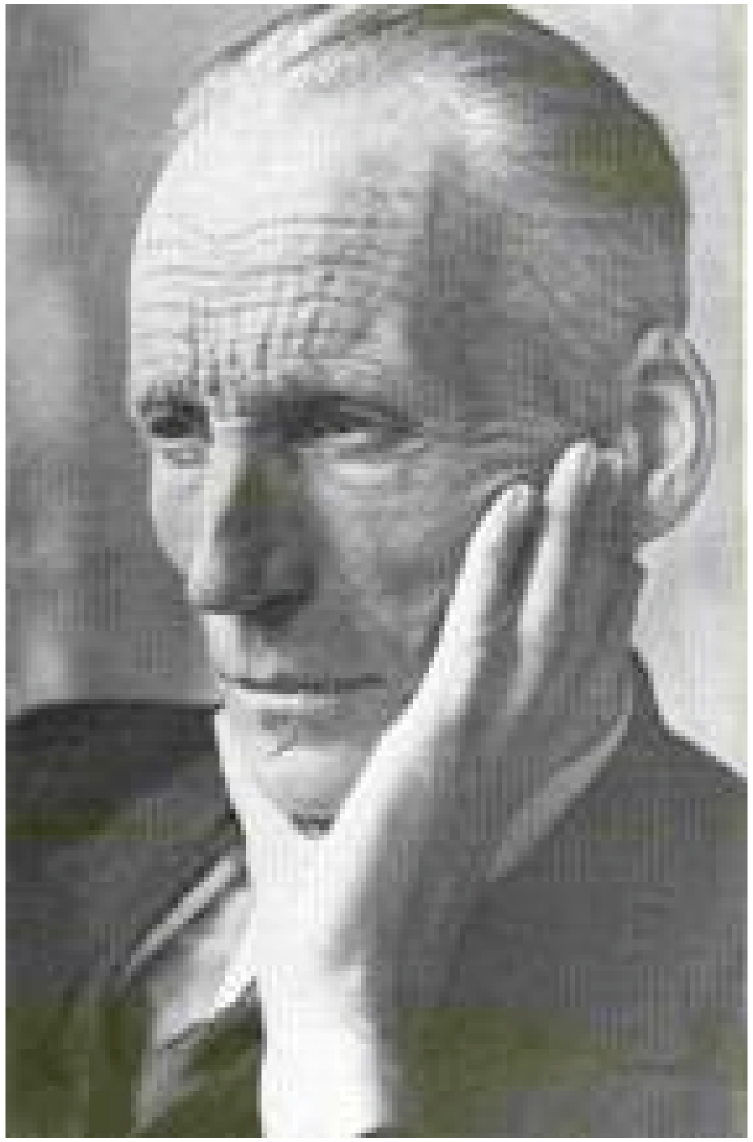
- Discuss Griss’ objection.
- Specific replies by Brouwer and Heyting.
- Explain why they fail to meet Griss’ challenge.

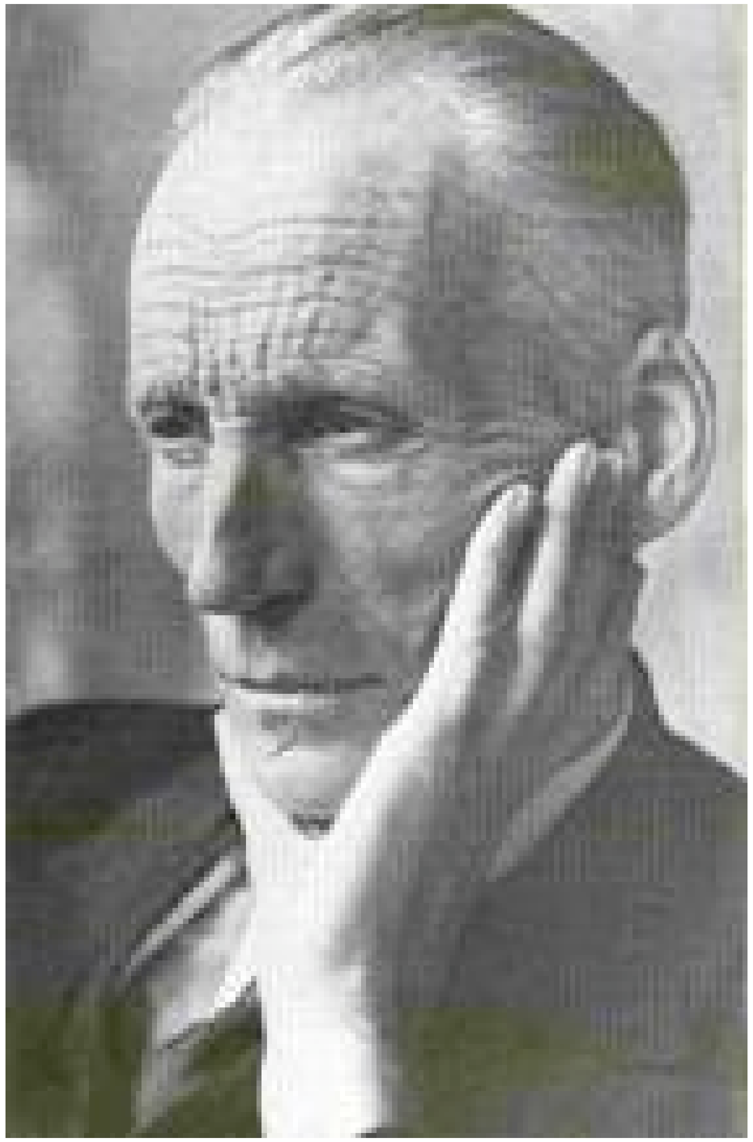
Brouwer's Intuitionism



Brouwer's Intuitionism

- Mathematics has...

  "its origin in the perception of a *move of time*, i.e. of the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the two-ity thus born is divested of all quality, there remains the *empty form of the common substratum of all two-ities*. It is this common substratum, this empty form, which is *the basic intuition of mathematics*."



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  - Logic and Language must take a back seat to Intuition.
    - They are vague and imprecise

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A

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A $\qquad$ B

Brouwer's Intuitionism

A

B

C

Brouwer’s Intuitionism

A *B*

D *C*

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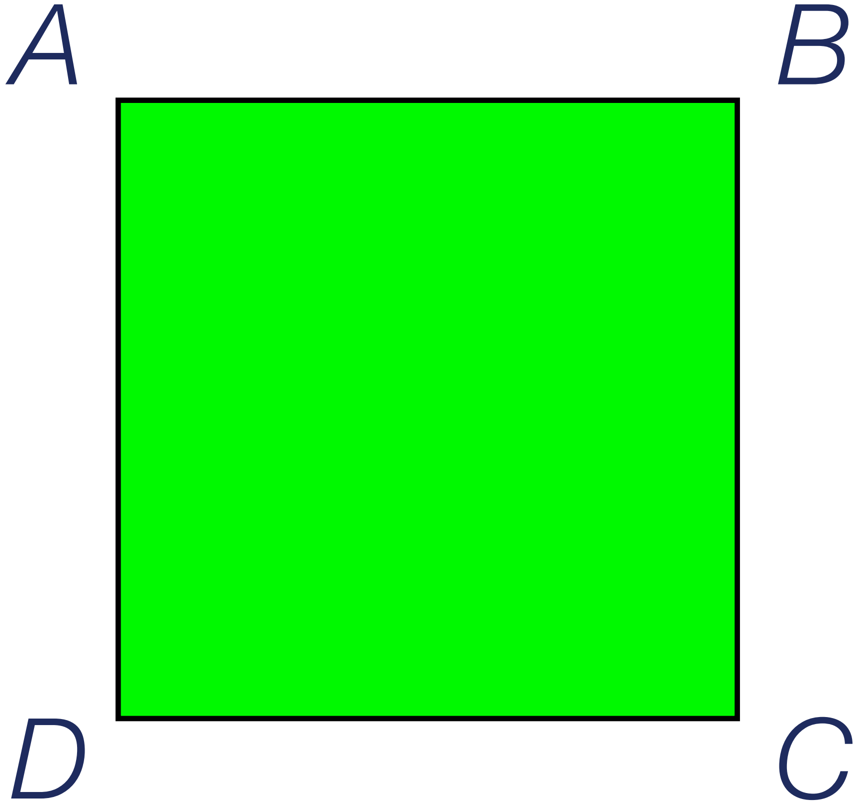

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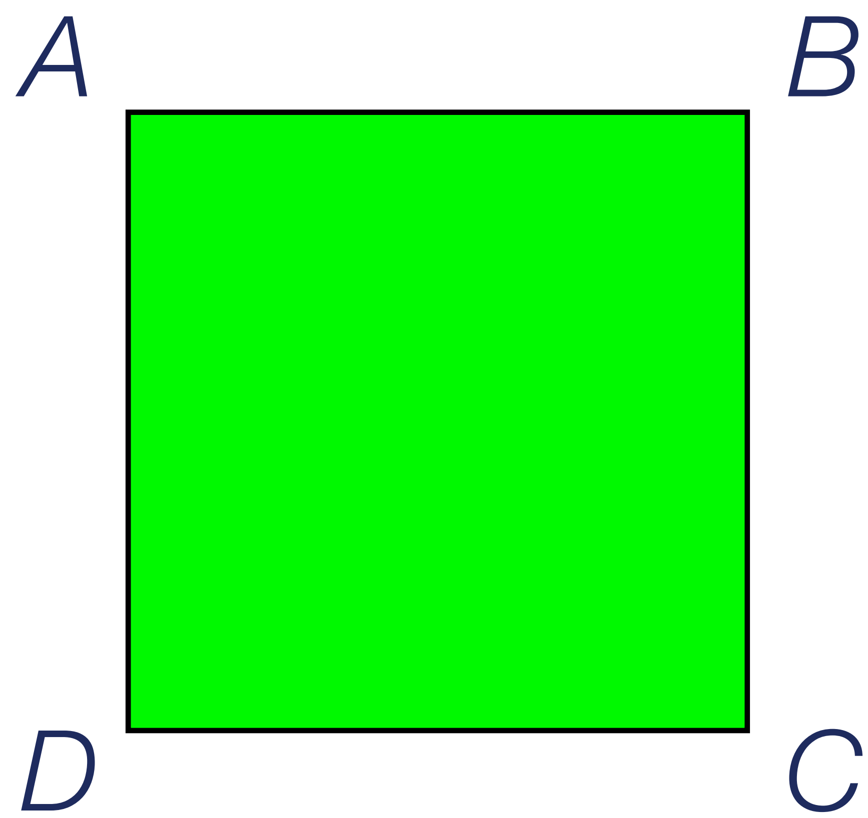

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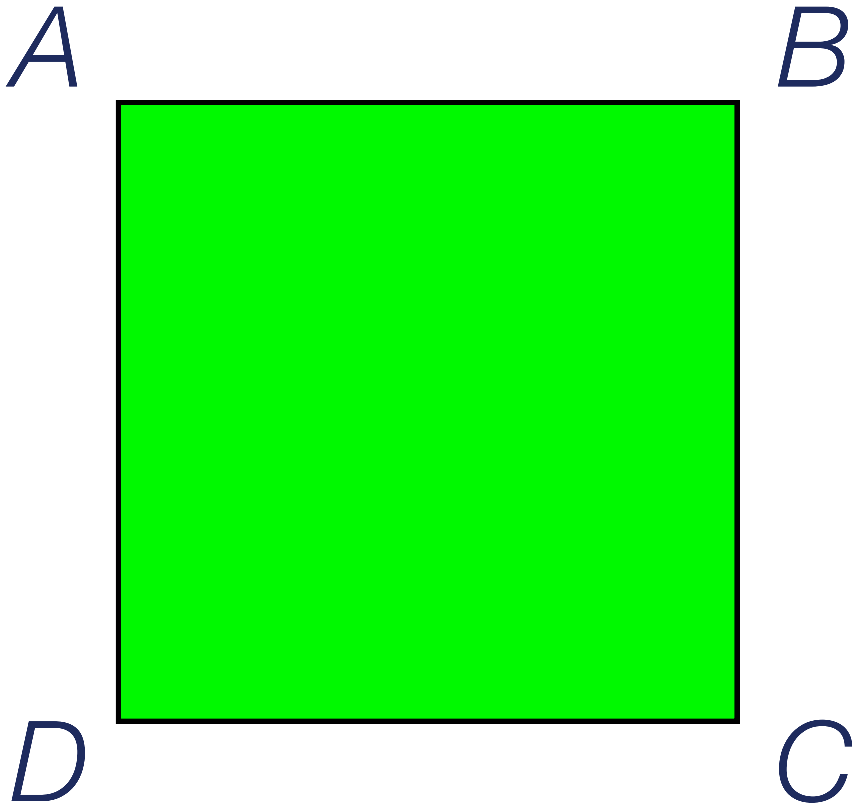

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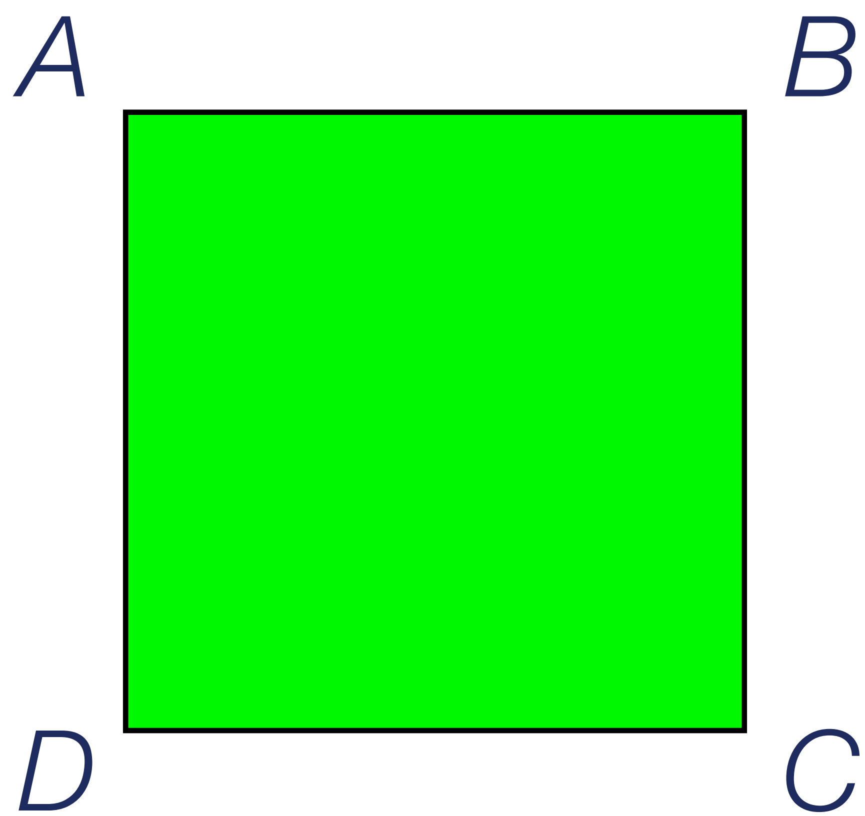

Circle

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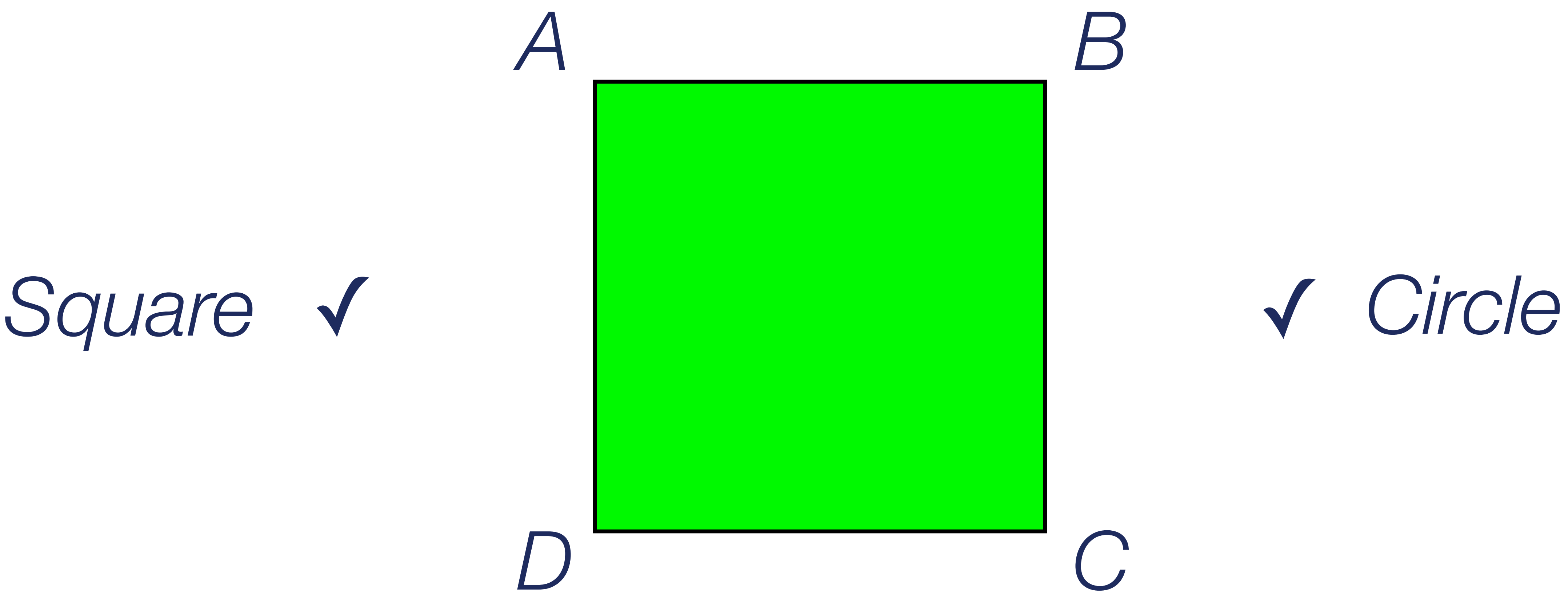

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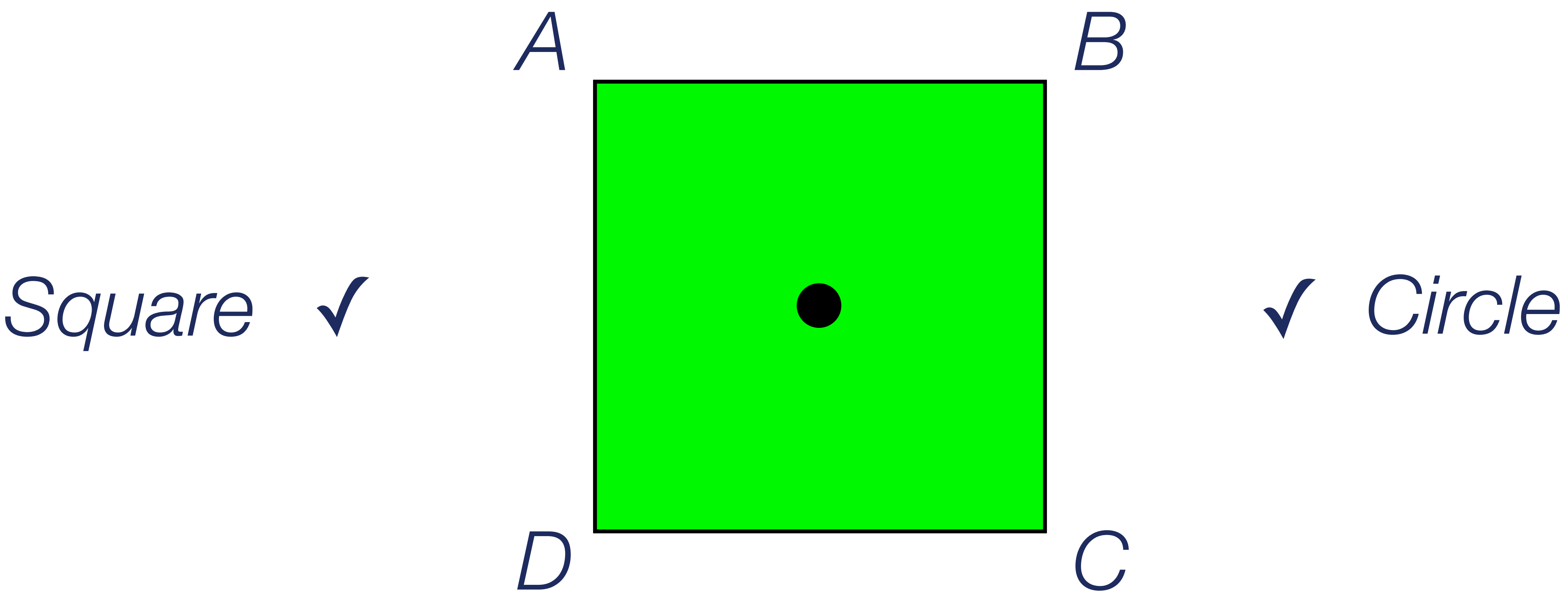

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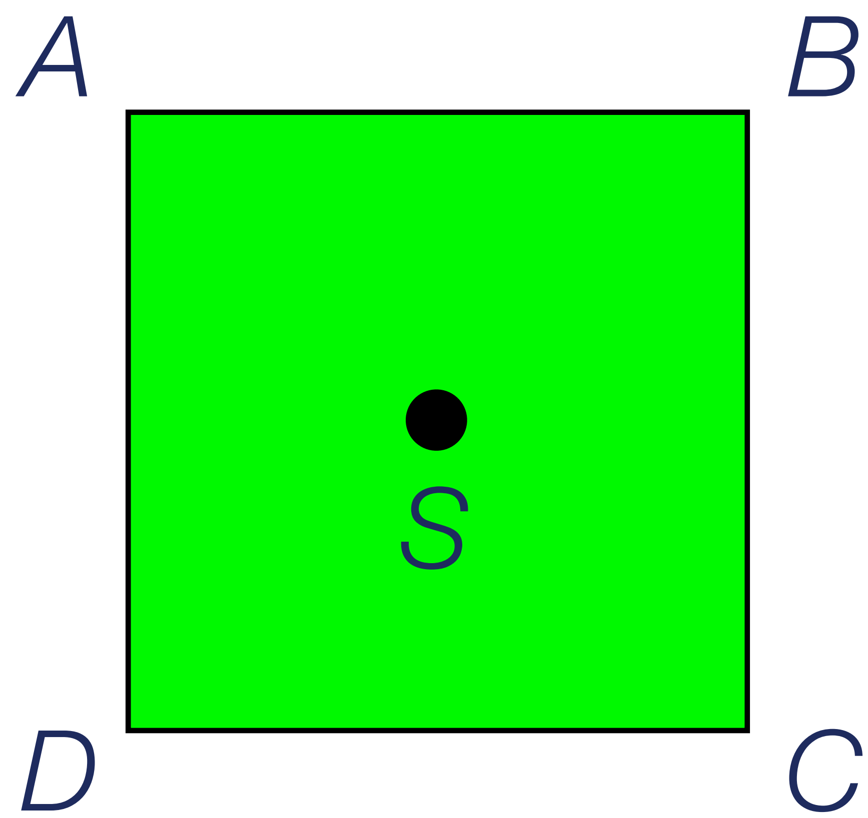

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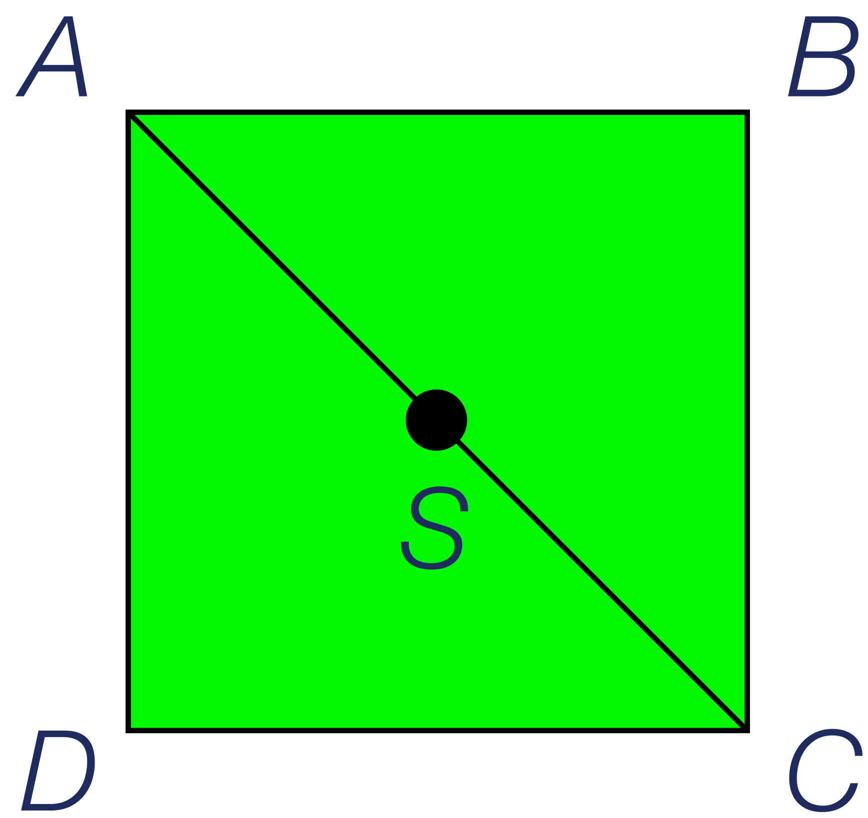

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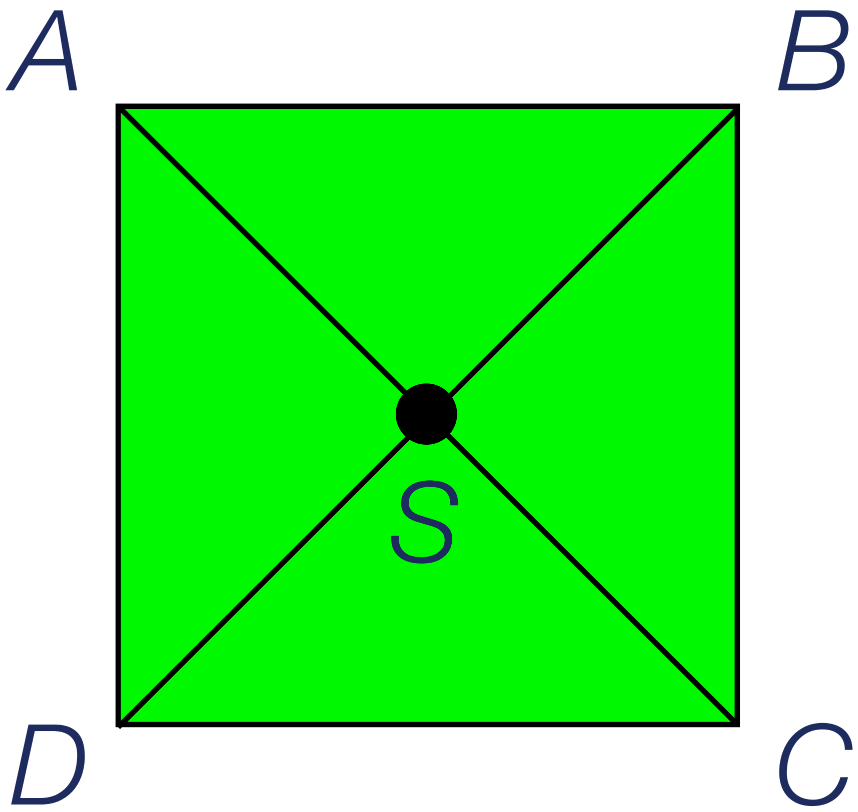

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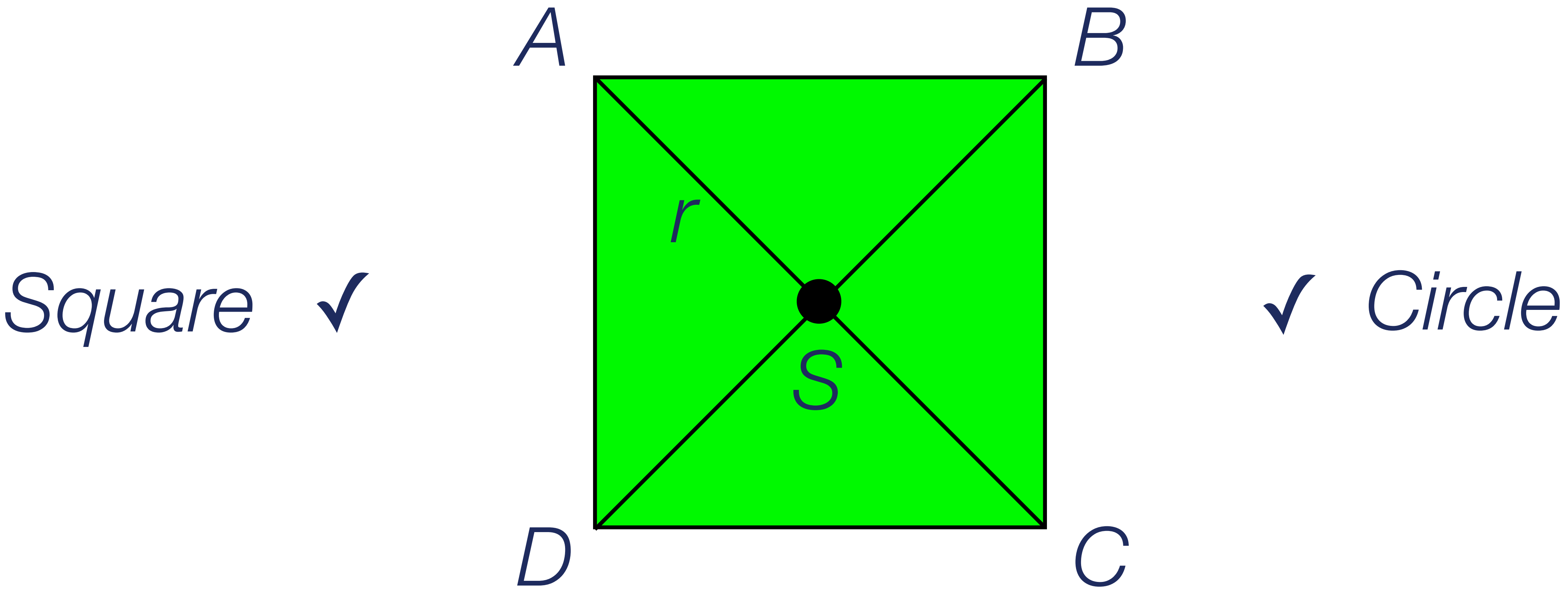

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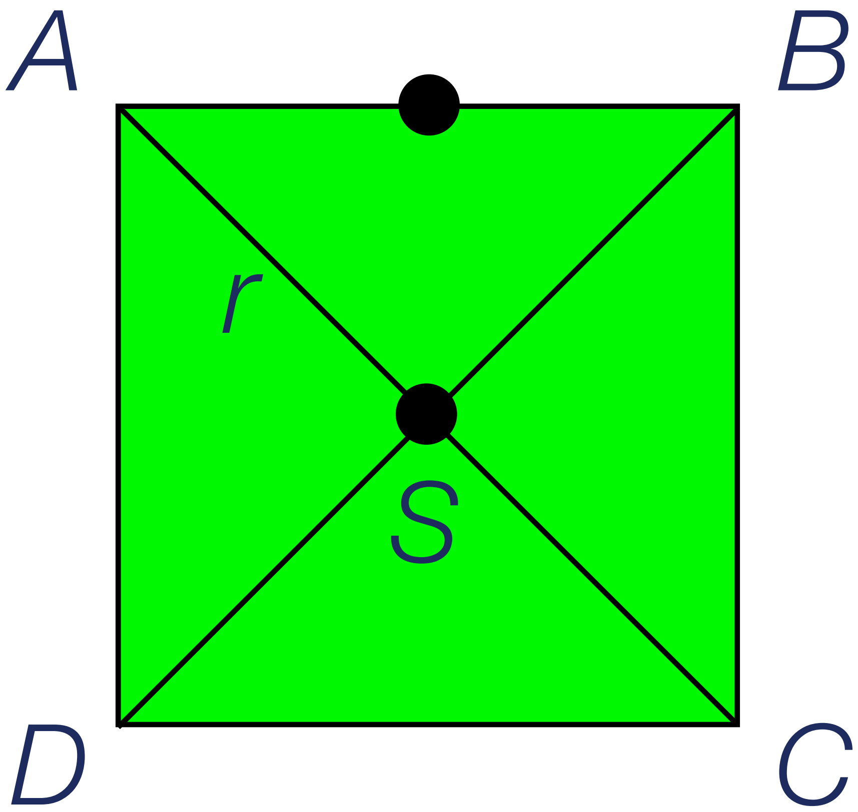

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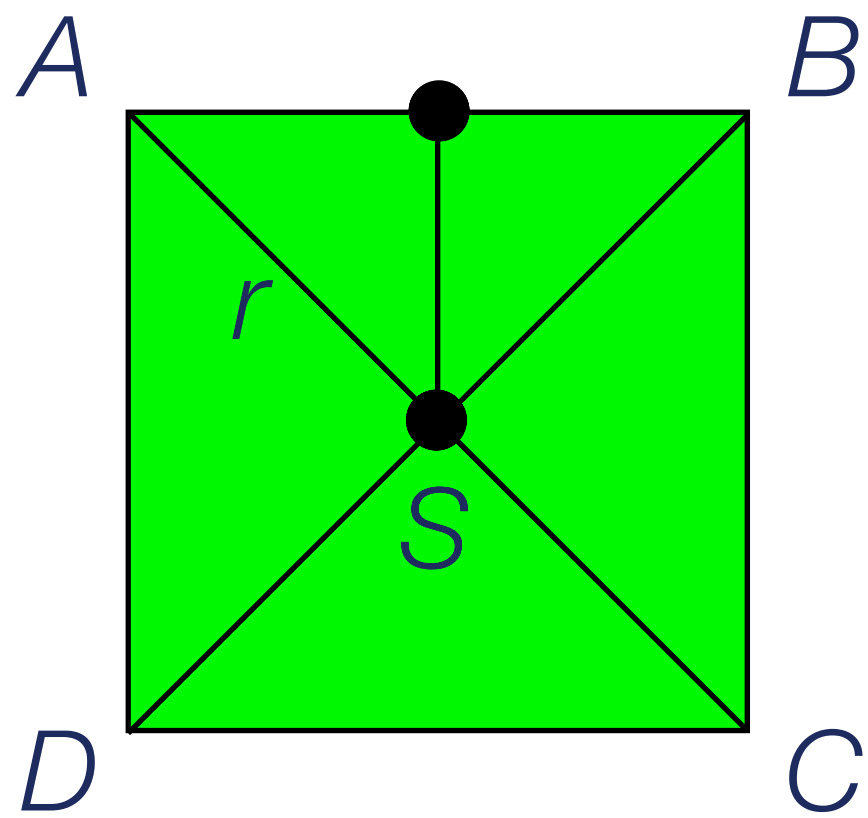

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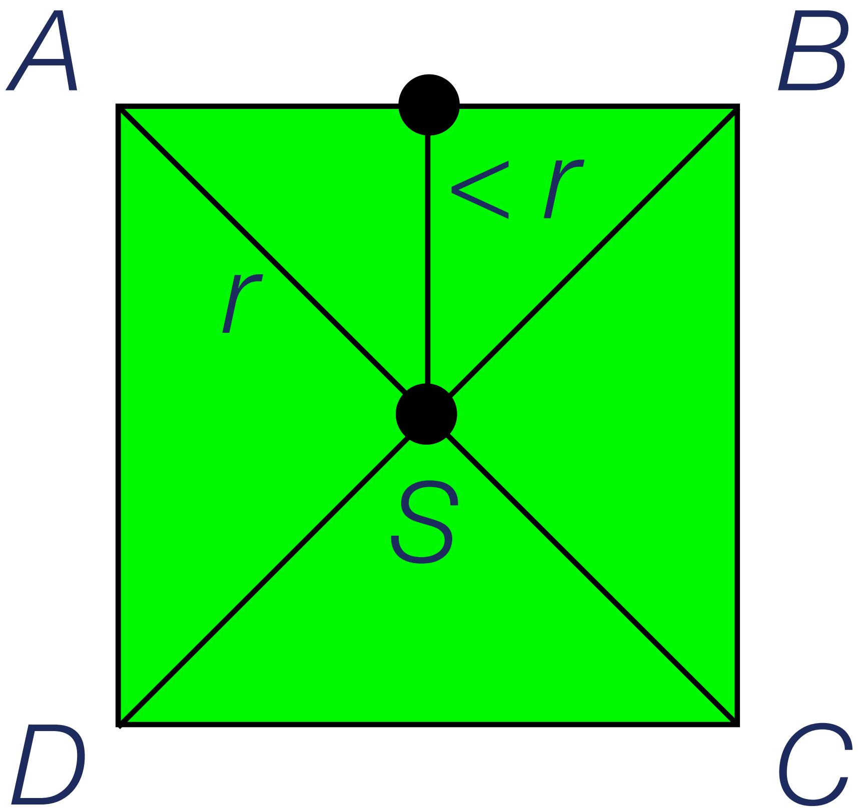

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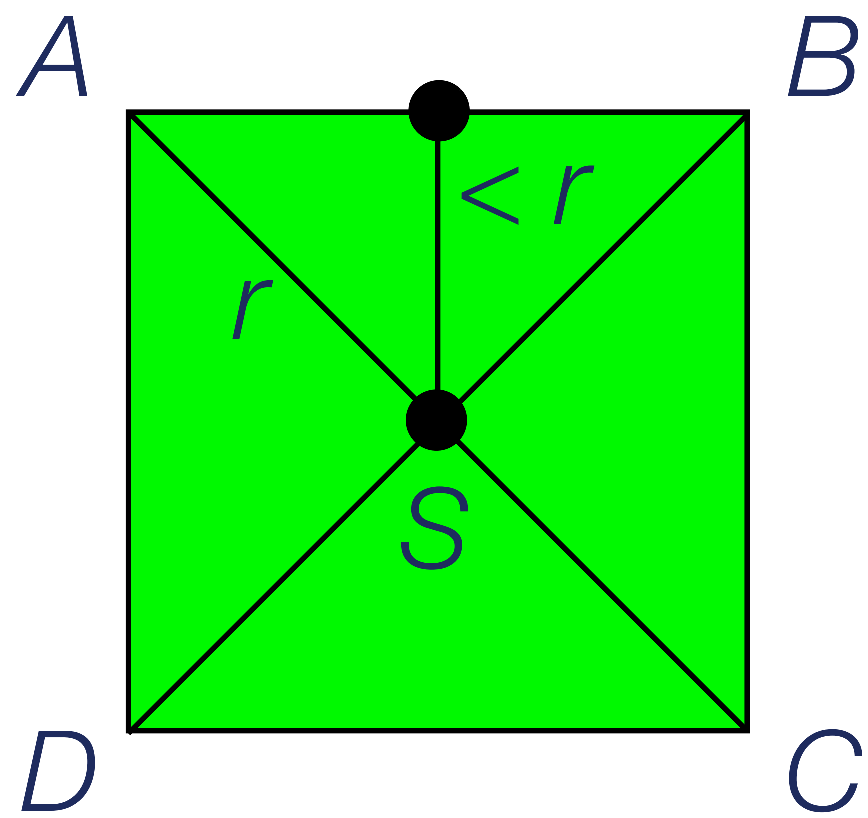

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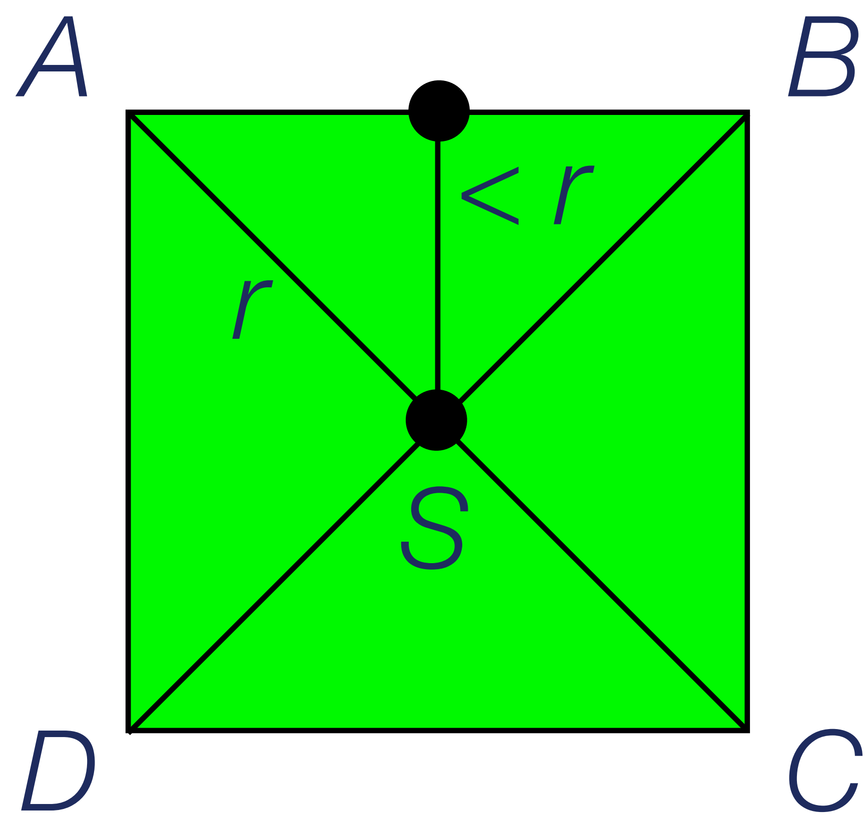

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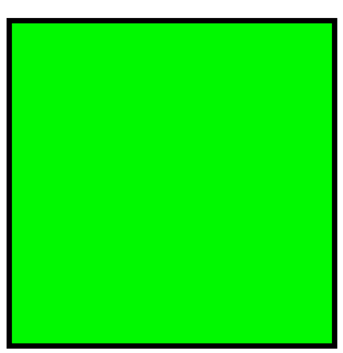

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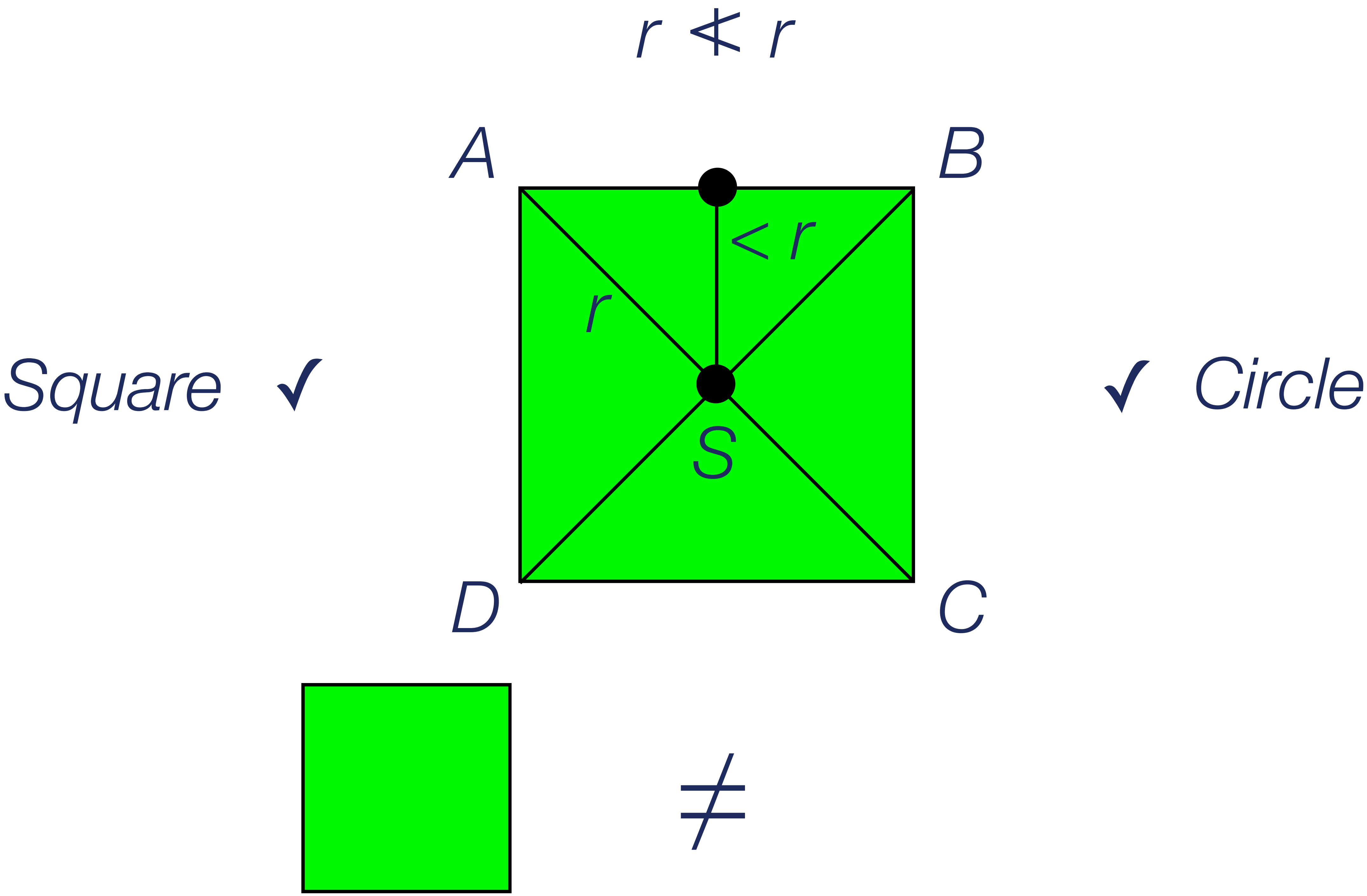

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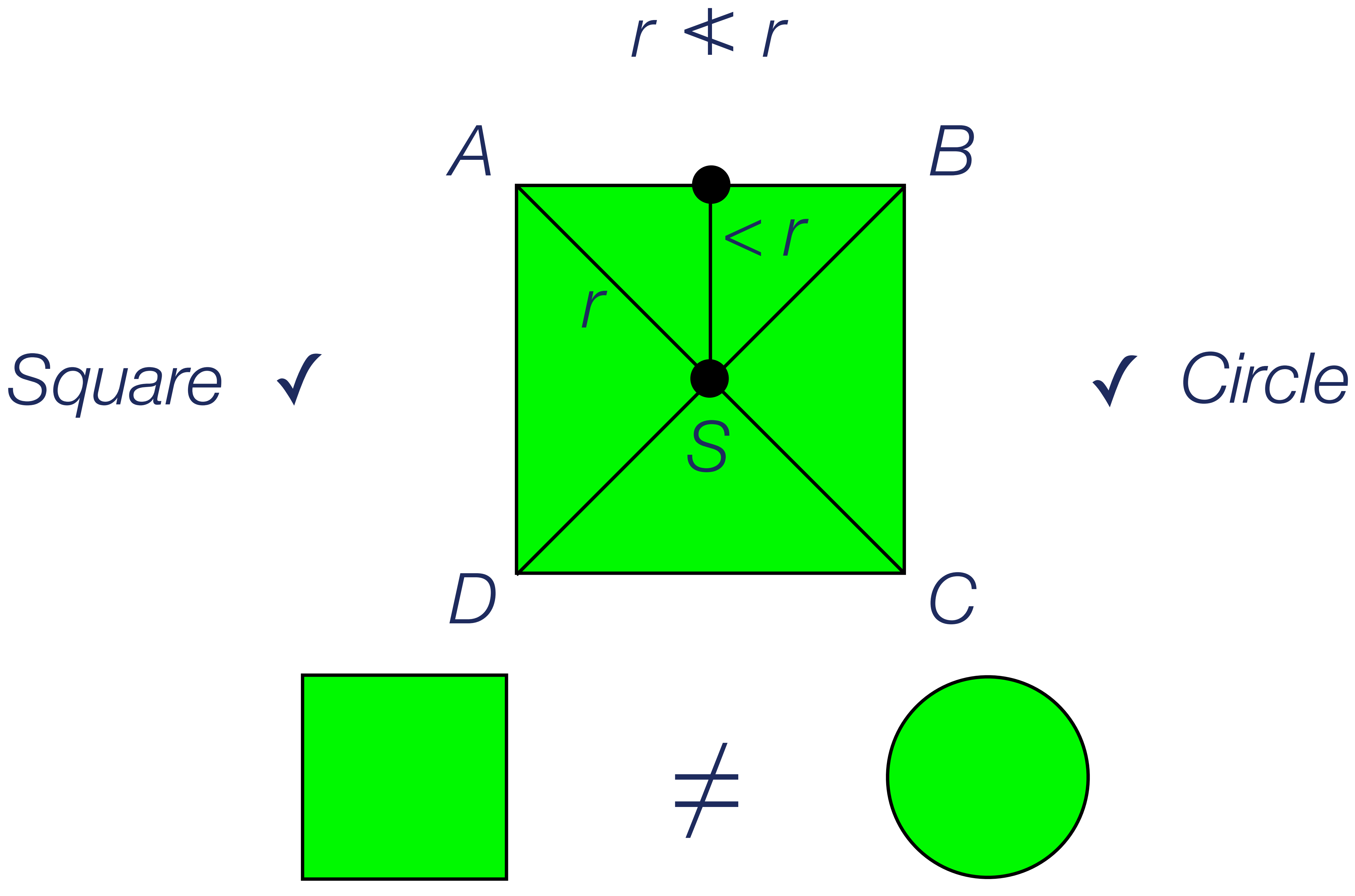

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- Argues mathematics should remain metaphysically neutral.
- Finds foundation in the immediately given perceptual and psychological faculties of the subject.
- Abstraction from the content of our individually distinguished perceptions gives rise to the notion of an infinitely proceeding sequence.
- How can we be justified in thinking our constructions are correct?

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  - Offers first formalization of intuitionistic logic - an *Assertability Calculus*.

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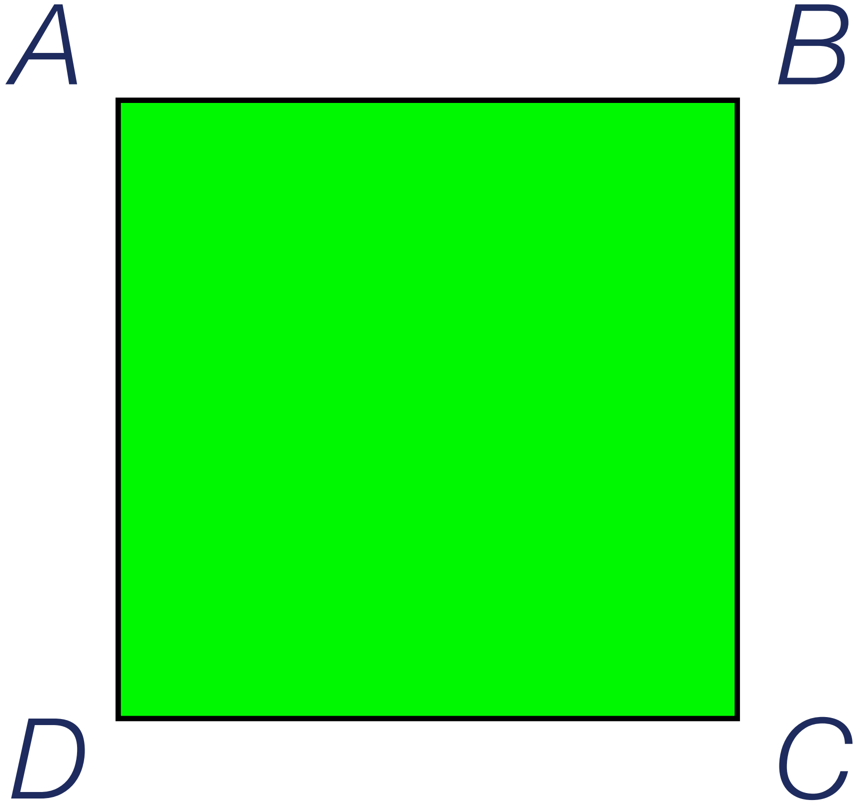

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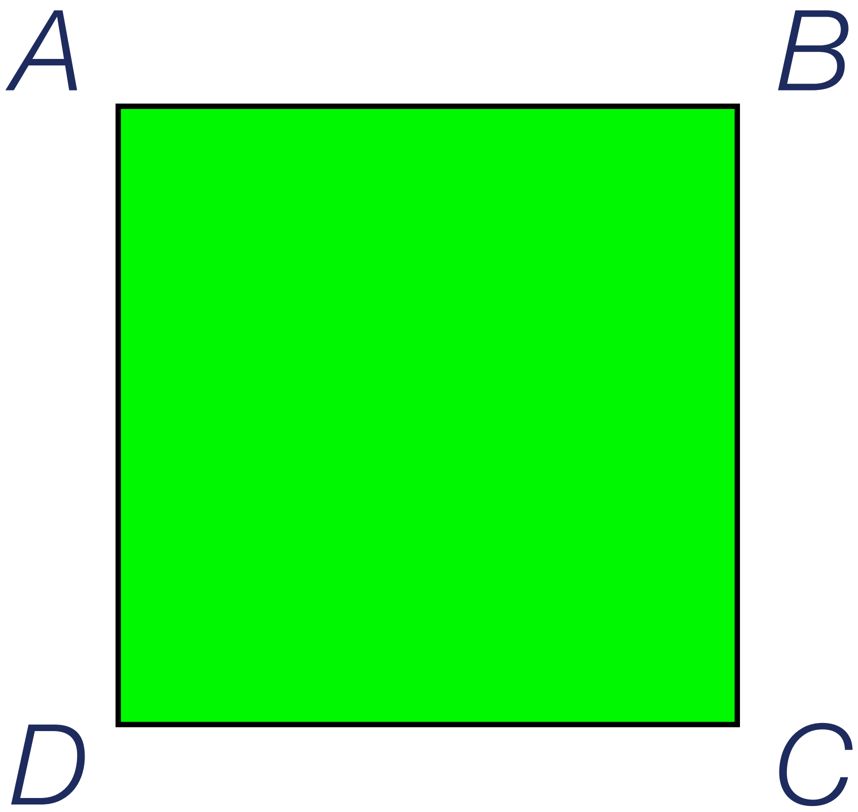

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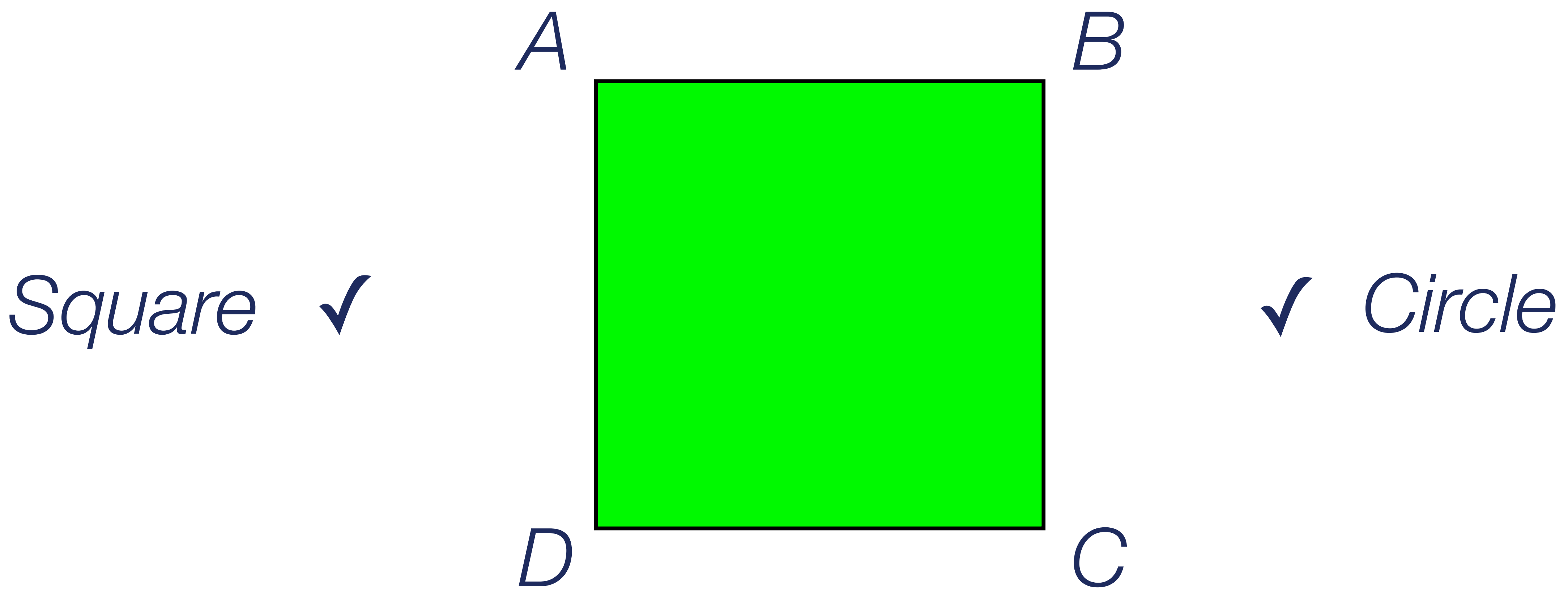

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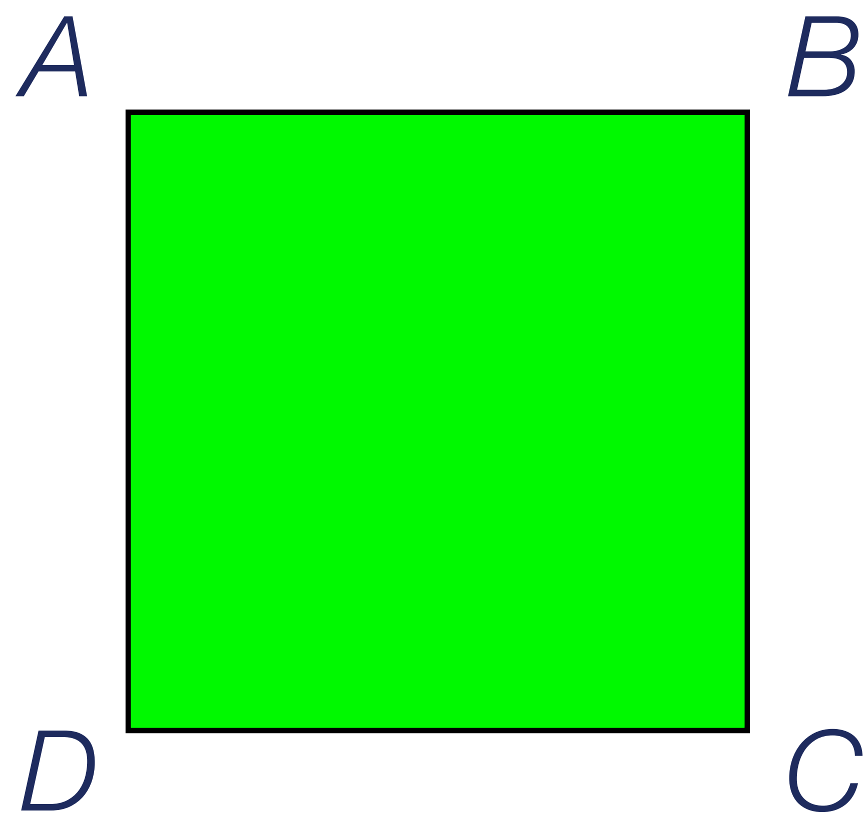

✓ Circle

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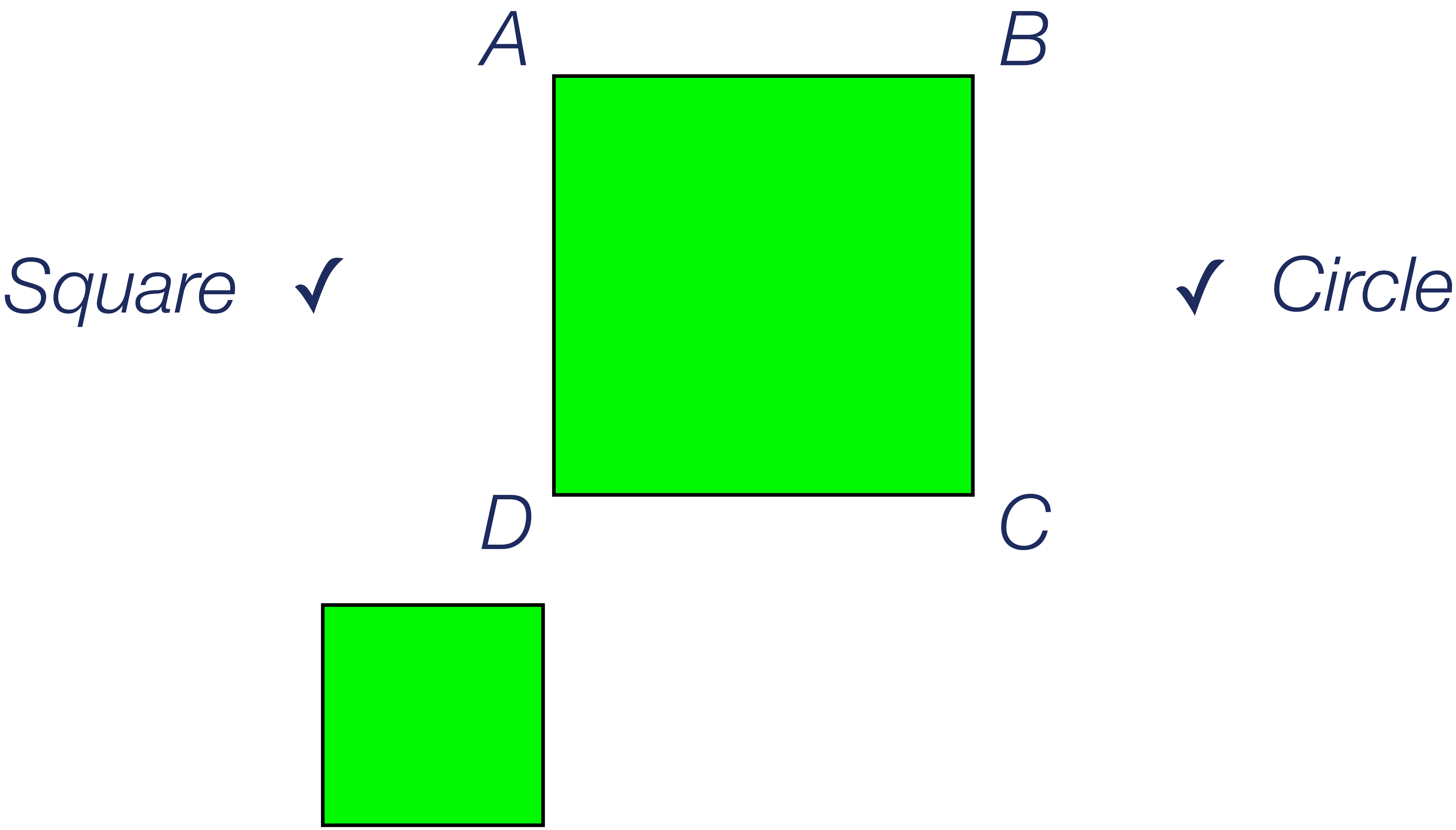

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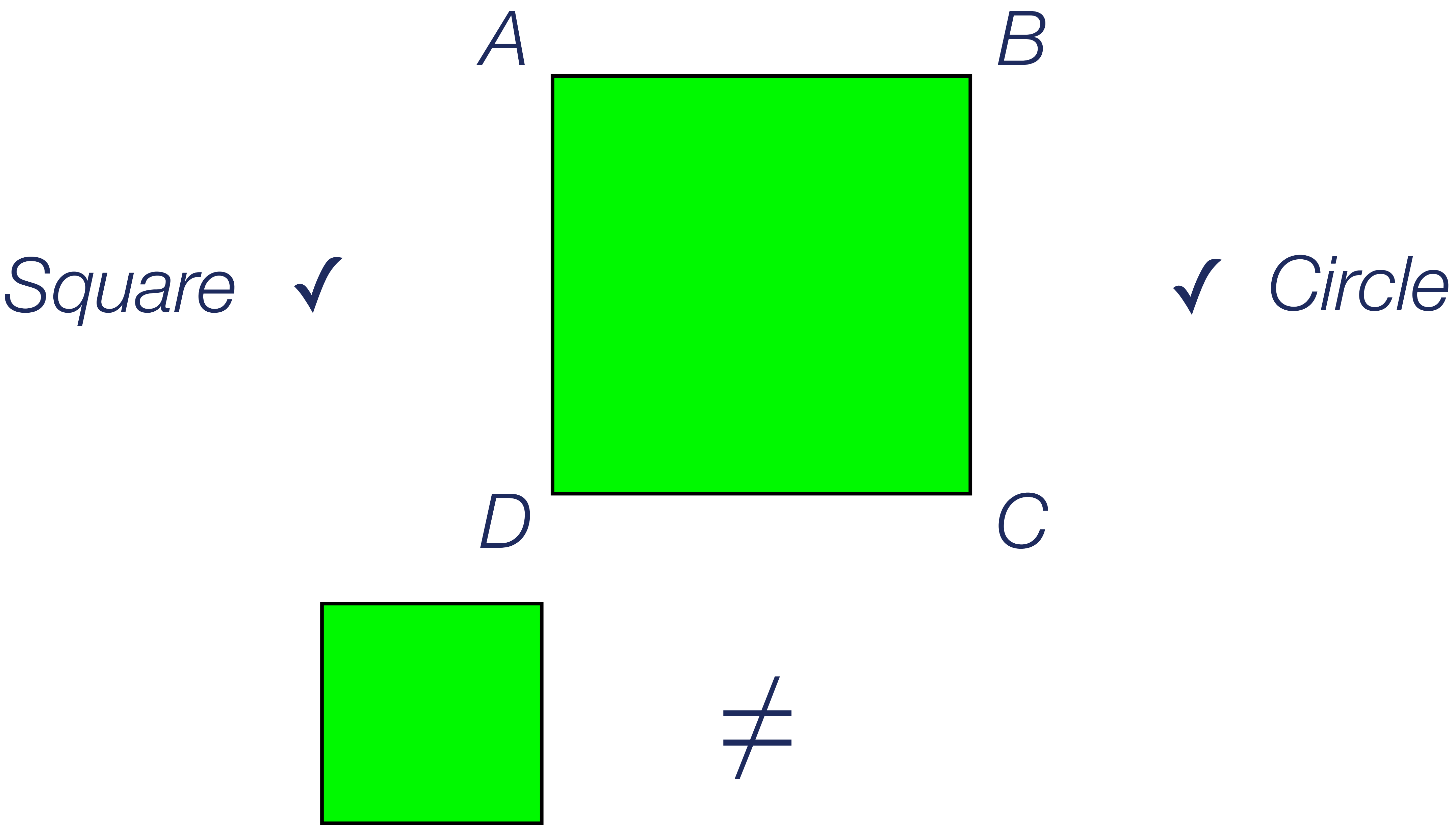

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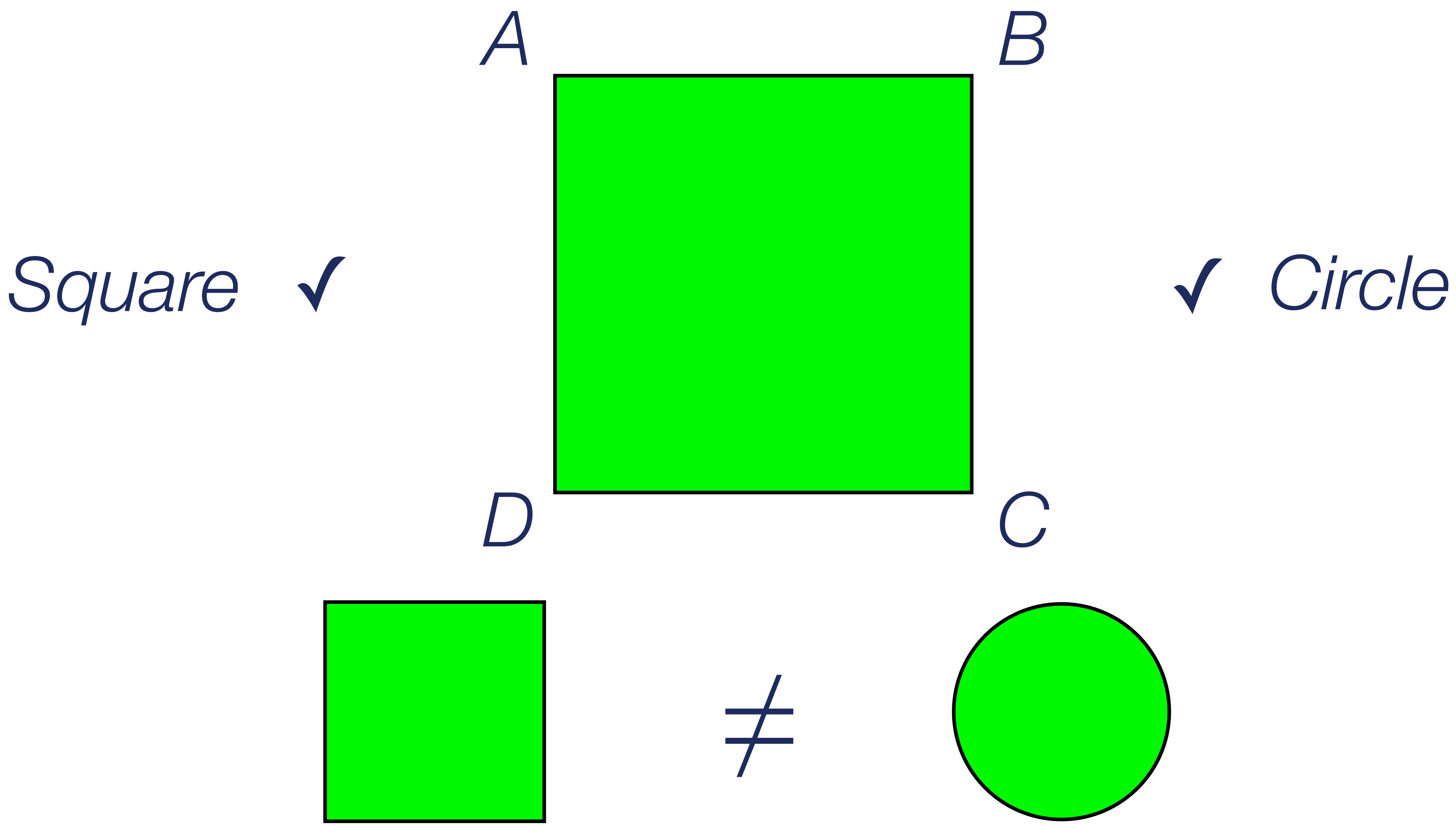

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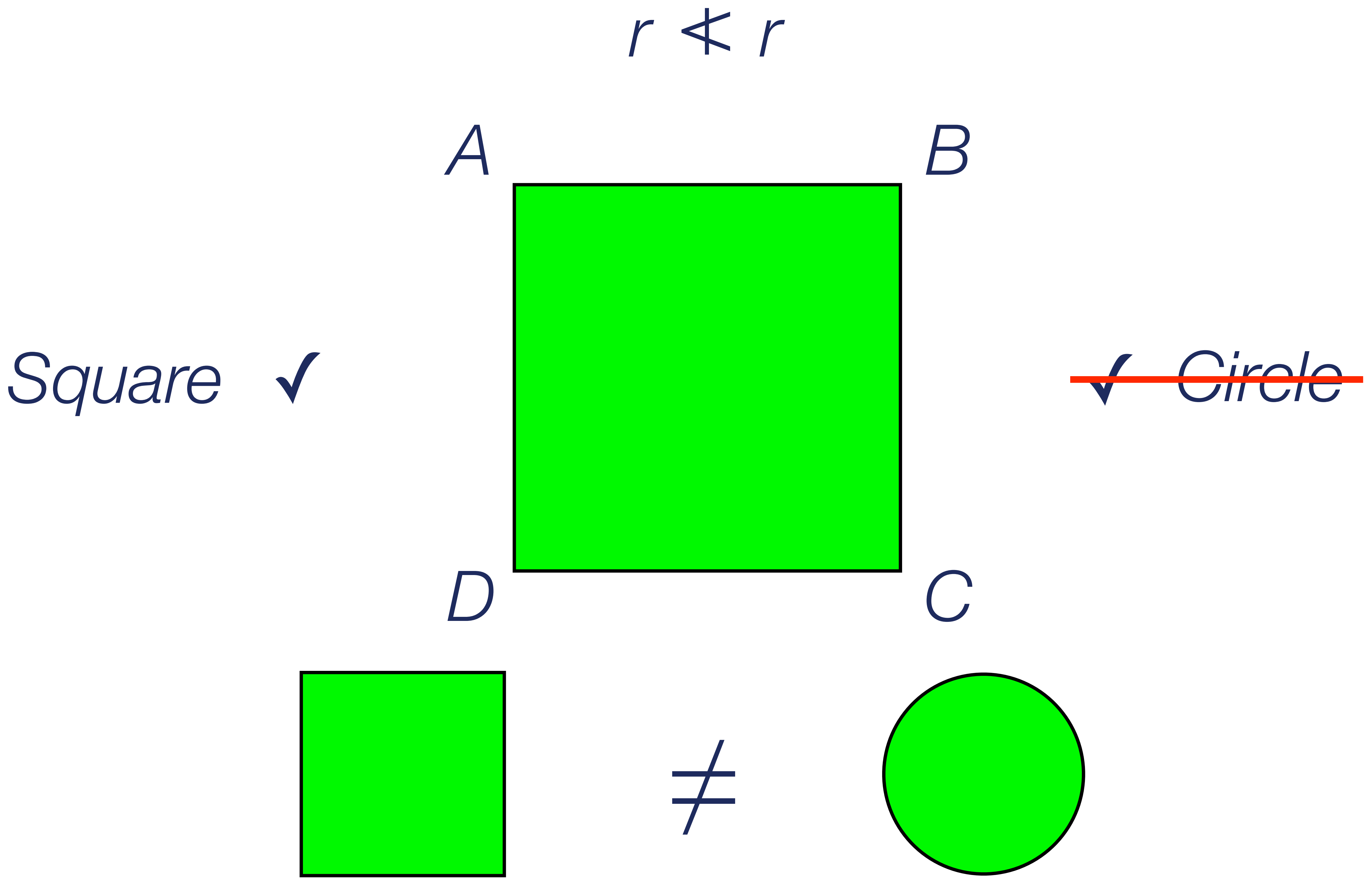

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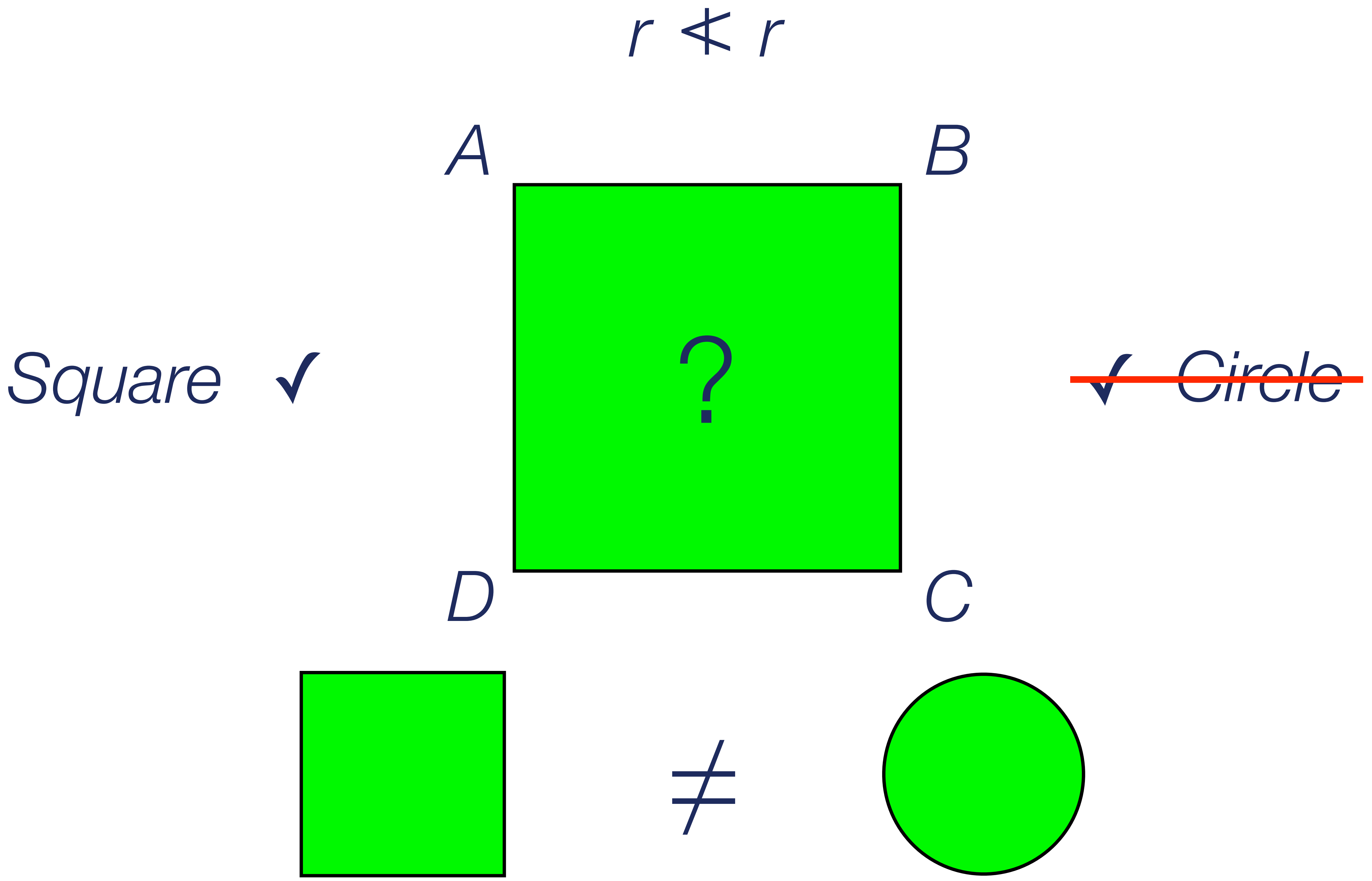

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  - '*x* is a square and *x* is a circle' is *nonsense*.

Brouwer’s Reply - Essentially Negative Properties

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- Brouwer counters by presenting a property essential to mathematics for which this cannot be done.
- The consequence for Griss would be the insecurity of a large portion of mathematics.

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- We may define a real number ρ, using a positively convergent sequence, as follows:

  Let α be an assertion such that it cannot currently be tested. For example, there is currently no known method to decide whether or not in the decimal expansion of π there occurs the 10-digit sequence 0123456789.

  One may then create an infinitely proceeding sequence of rational numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$, according to the following rule:

  In the course of choosing a_n, $a_n = 0$ if α cannot be decided; a_n, and for every natural number v, $a_{n+v} = 2^{-n}$ if a proof for α is discovered; a_n, and for every natural number v, $a_{n+v} = -2^{-n}$ if the absurdity of α is discovered.

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  Alternatively, if $\rho < 0$, then $\rho > 0$ would be impossible, and so it would be certain that the assertion α could not be proven true. Similar to before, the absurdity of α (i.e. $\sim\alpha$) would be known, and therefore α would be tested, which it is not. *Therefore,* $\rho < 0$ *does not hold.*

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  Suppose finally that $\rho = 0$. Then neither $\rho < 0$ nor $\rho > 0$ could be shown, and so neither the truth nor the absurdity of α could be proved. The consequence of this is that both the absurdity α and the absurdity of the absurdity of α would be known, which is a contradiction. *Therefore, $\rho = 0$ is absurd* (i.e. $\rho \neq 0$).

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  Consequently, for the real numbers ρ and 0, the negative property $\rho \neq 0$ holds, while neither $\rho > 0$ nor $\rho < 0$ is present. Therefore, $\rho \neq 0$ is demonstrated as *essentially negative*.

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- Clearly, this is not so straightforward with regard to the reals.
- Our rule guarantees, that while we know the value of $\rho \neq 0$, we know neither that $\rho > 0$ nor that $\rho < 0$.
- Thus we must conclude that $\rho \neq 0$ without being able to translate this into an equivalent positive relation.

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- Still, Brouwer's contention that *Difference* is essentially negative is called into question.

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- This notion of '*Stages of Evidence*' can be paired with the argument that...

  "It seems to me that it is impossible to banish all unrealized suppositions from mathematics, for such suppositions are implicit in every general proposition. If we say "*for all* real numbers a and b, $a + b = b + a$", this means the same as "*if we construct* two real numbers a and b, then $a + b = b + a$", but we have not actually constructed them. If we wish to avoid altogether the unrealized suppositions, we must reckon all such general formulas to pre-mathematical reasoning, while only the concrete cases such as $2 + 3 = 3 + 2$ would belong to mathematics."

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- Thus, the adoption of Griss' confined notion of construction leads to the banishment of all *general* forms of reasoning.

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Conclusions

- We have seen that the intuitionist trades in entities constructed by the mind itself.

- Thus mathematics can admit only what can be constructed.

- By conceiving of negation as an assertion that implies a contradiction, the intuitionist seems to fall into Griss' difficulty.

- While we might have to surrender a large portion of mathematics, the intuitionist seems to have little choice.

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Thank You.